Ways of Estimating Extreme Percentiles for Capital Purposes

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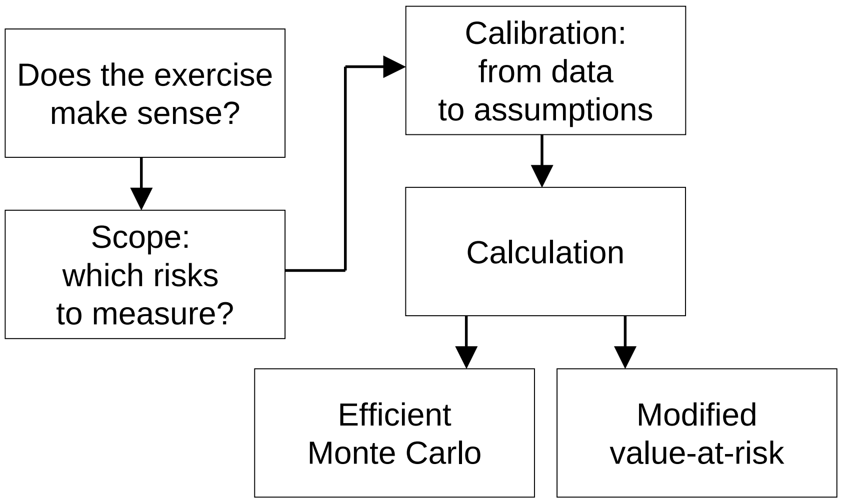
This is the framework we're discussing

Assessing Capital based on:

•Projected Assets > Liabilities
•In one year
•With very high (eg 99.9%) probability
•Applies to life and property-casualty

Decision Path



Value at Risk (VaR)

Value at Risk – Market Level Assumptions

Driver	standard deviation	correlations											
Equity	20%	100%											
Property	15%	50%	100%										
Yield Curve	0.80%	0%	-30%	100%									
Credit Spread	0.30%	-50%	-40%	10%	100%								
Property loss ratio	10.00%	0%	30%	-10%	25%	100%							
Liability loss ratio	15.00%	20%	10%	-15%	25%	50%	100%						
Inflation	1.00%	-50%	-40%	40%	20%	0%	0%	100%					
Mortality	35%	0%	0%	0%	0%	0%	0%	0%	100%				
Lapses	35%	-30%	-30%	30%	0%	-30%	-30%	20%	0%	100%			
Operational	100%	30%	30%	-30%	-20%	-30%	-30%	20%	0%	40%	100%		
Liquidity	100%	25%	25%	-25%	-50%	-20%	-20%	10%	10%	25%	10%	100%	
Group	100%	20%	20%	-20%	-20%	-20%	-20%	20%	20%	20%	20%	20%	100%

- Bank VaR typically 200 × 200 correlation matrix

Fixing the Correlation Matrix

correlations

100%											
50%	100%										
0%	-30%	100%									
-50%	-40%	10%	100%								
0%	30%	-10%	25%	100%							
20%	10%	-15%	25%	50%	100%						
-50%	-40%	40%	20%	0%	0%	100%					
0%	0%	0%	0%	0%	0%	0%	100%				
-30%	-30%	30%	0%	-30%	-30%	20%	0%	100%			
30%	30%	-30%	-20%	-30%	-30%	20%	0%	40%	100%		
25%	25%	-25%	-50%	-20%	-20%	10%	10%	25%	10%	100%	
20%	20%	-20%	-20%	-20%	-20%	20%	20%	20%	20%	20%	100%

Not sufficient to have correlations between ± 100%.
Only positive definite matrices can be valid correlation matrices

best fit positive definite correlations

100%											
51%	100%										
-4%	-31%	100%									
-49%	-40%	9%	100%								
-1%	29%	-9%	25%	100%							
18%	9%	-13%	25%	50%	100%						
-44%	-37%	35%	20%	-1%	-2%	100%					
1%	0%	-1%	0%	0%	0%	1%	100%				
-26%	-28%	26%	0%	-30%	-31%	23%	0%	100%			
25%	27%	-25%	-20%	-28%	-27%	15%	-1%	35%	100%		
22%	23%	-22%	-50%	-19%	-18%	7%	10%	22%	12%	100%	
17%	19%	-17%	-20%	-19%	-19%	17%	20%	18%	22%	21%	100%

The larger the matrix, the more likely it is that positive definiteness is a problem.

Calculating Value at Risk

Test	Stress	Free Assets	Beta	Capital required
Base Case		200		
Equity	-40%	170	75	39
Property	-25%	183	68	26
Yield Curve	1%	185	-1500	31
Credit Spread	1%	185	-1500	12
Property loss ratio	20%	175	-125	32
Liability loss ratio	20%	170	-150	58
Inflation	1%	180	-2000	52
Mortality	40%	190	-25	23
Lapses	40%	195	-12.5	11
Operational	-1	190	10	26
Liquidity	-1	195	5	13
Group	-1	195	5	13
Total				334
Diversification credit				184
Net required				150

Room for Improvement?

- VaR runs instantly and parameters / assumptions are transparent
- Non-zero mean
  - easy to fix
  - take credit for one year's equity risk premium or one year's profit margin in premiums
- Path dependency, overlapping cohorts
  - Add more variables, which can result in huge matrices to estimate
- Company depends linearly on drivers
  - mitigate by careful choice of stress tests
  - worst for GI because of reinsurance
  - may need mini DFA model to calibrate a VaR model
- Multivariate Normality
  - Strong assumption – was often supposed lethal
  - Before we understood large deviation theory

Large Deviation Theory

Large Deviation Expansions

- In many important examples, we can estimate the moment generating function of net assets
- Large deviation expansions are an efficient way to generate approximate percentiles given moment generating functions
- Exact formulas do exist but they involve numerical integration of complex numbers

LD Expansion: The Formula

To estimate **Prob**{$X \le c$}
Where **E**exp(pX) = exp[κ(p)]
Find p where κ'(p)=c

$$\eta_0 = p \times \sqrt{\frac{2\kappa'(p)}{p} - \frac{2\kappa(p)}{p^2}}$$

$$\eta_1 = \frac{1}{\eta_0} \ln\left[\frac{p\sqrt{\kappa''(p)}}{\eta_0}\right]$$

$$\eta_2 = \ldots$$

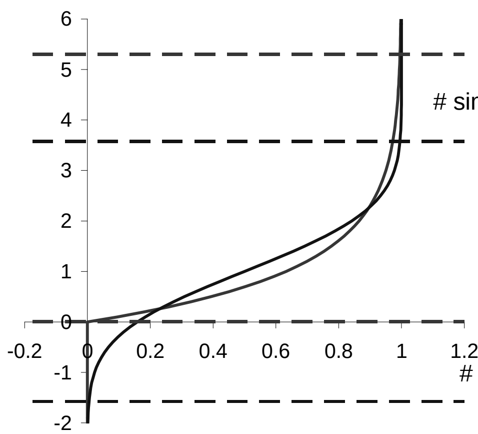
$$\mathbf{Prob} \sim \Phi(\eta_0 + \eta_1 + \eta_2 + \ldots)$$

Φ = cumulative normal function

Try X ~ normal(μ,σ^2)
$\kappa(p) = \mu p + \frac{1}{2}\sigma^2 p^2$
$\kappa'(p) = \mu + \sigma^2 p$
$p = \sigma^{-2}(c-p)$
$\eta_0 = \sigma^{-1}(c-p)$
$\eta_1 = 0$
LD expansion exact

Try X ~ exponential (mean 1)
Eexp(pX) = $(1-p)^{-1}$
$\kappa(p) = -\ln(1-p)$
$\kappa'(p) = (1-p)^{-1}$
$p = 1-c^{-1}$
$\kappa''(p) = (1-p)^{-2}$

Comparison $\eta_0+\eta_1$ with Monte Carlo



LD expansion	5.3221
exact 99.5%-ile	5.2983
# sims for same error	350 000
normal 99.5%-ile	3.5758
LD expansion	0.0048
exact 0.5%-ile	0.0050
# sims for same error	180 000
normal 0.5%-ile	-1.5758

blue = normal(1,1) red = exponential

LD Expansion Needs Analytical MGF

Easy

- Normal
- Gamma
- Inverse Gaussian
- Reciprocal Inverse Gaussian
- Generalised hyperbolic
- Poisson / Neg Binomial compounds of the above
- Mixtures of the above

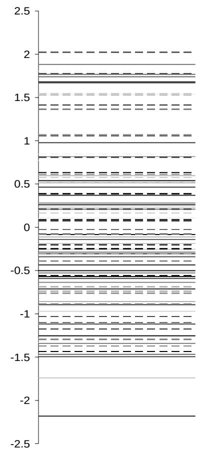
Tricky

- Pareto
- Lognormal
- Weibull
- Copula approaches

Key question: Is there sufficient data to demonstrate we have a tricky problem?

Efficient Simulations: Importance Sampling

Importance Sampling – How it Works

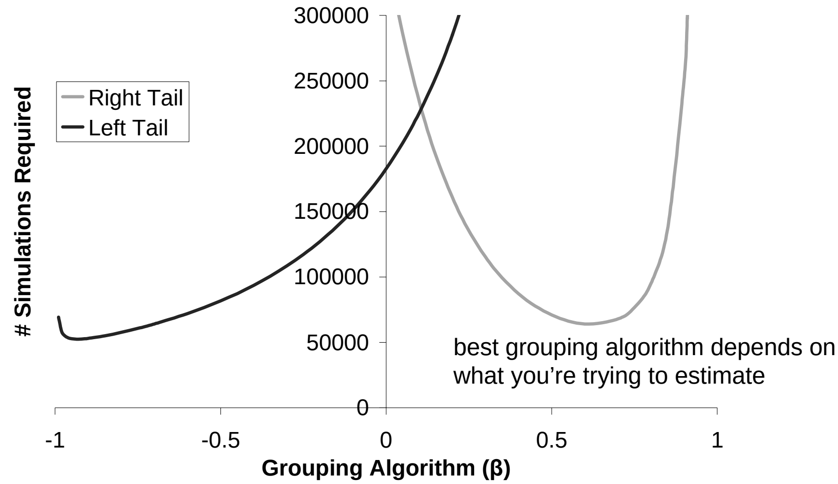


- Generate 1 000 000 simulations
- Group into 1 000 model points
- Outliers: treat individually
- Near the centre: groups of 5000 observations or more for each model point
- Result: 1 000 model points with as much information as 20 000 independent simulations

Importance Sampling: Another View

- We wish to simulate from an exp(1) distribution
  – density $f(x) = \exp(-x)$
- Instead simulate for an exp(1-β) distribution
  – density $g(x) = (1-\beta)\exp[-(1-\beta)x]$
  – weight $w(X) = (1-\beta)^{-1}\exp(-\beta X)$
- Use weighted average to calculate statistics
  – equivalent to grouping (yes it *does* work!)
- Product rule for multiple drivers

Effectiveness compared to LD



Testing Extreme Value Calibrations

Extreme Value Theory

Central Limit

- If X_1, X_2, X_3 … X_n are i.i.d.
- Finite mean and variance
- Then the average A_n is asymptotically normal
- Useful theorem because many distributions are covered
- Often need higher terms (eg LD expansion).

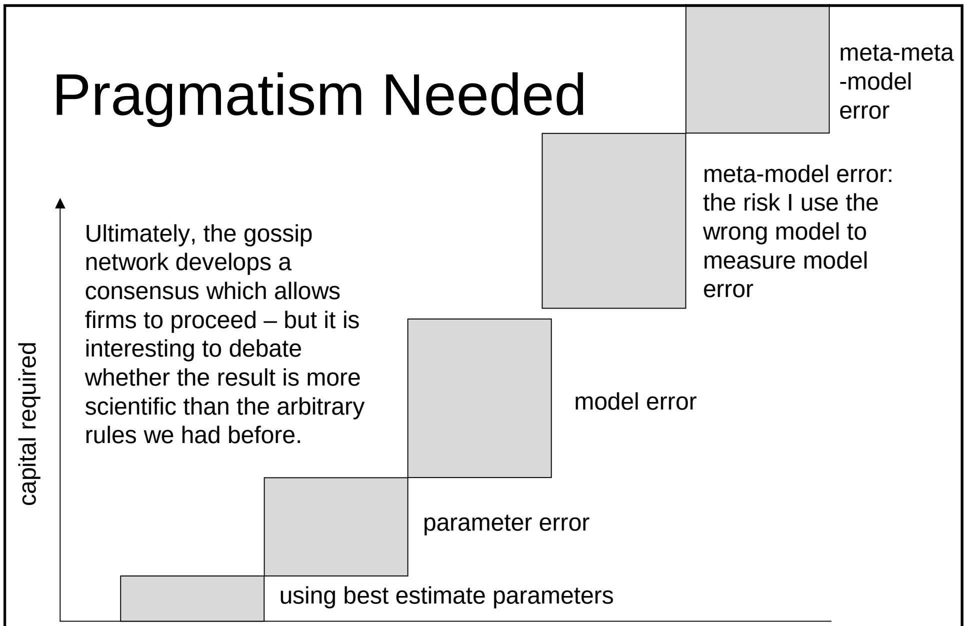
Extreme Value

- If X has an exponential / Pareto tail
- Then (X-k|X>k) has an asymptotic exponential / Pareto distribution
- Many distributions have no limit at all
- Higher terms in the expansion poorly understood

Estimating Extreme Percentiles

- Suppose "true" distribution is lognormal with parameters $\mu=0$, $\sigma^2=1$.
- Simulate for 20 years
- Fit extreme value distribution to worst 10 observations
- Don't need to calculate to see this isn't going to work
- Instability and bias in estimate of 99.5%-ile
- The extreme event: if you have one in the data set its over-represented, otherwise its under-represented.
- Conclusion is invariably a judgment call – was 11/09/2001 a 1-in-10 or 1-in-500 event? What's the worst loss I ever had / worst I can imagine – call that 1-in-75.
- Problems even worse when trying to estimate correlations / tail correlations / copulas
- Reason to choose a simple model with transparent inputs

Pragmatism Needed



Scope – Which Risks to Measure?

Apocalyptic Events

Capital partially effective

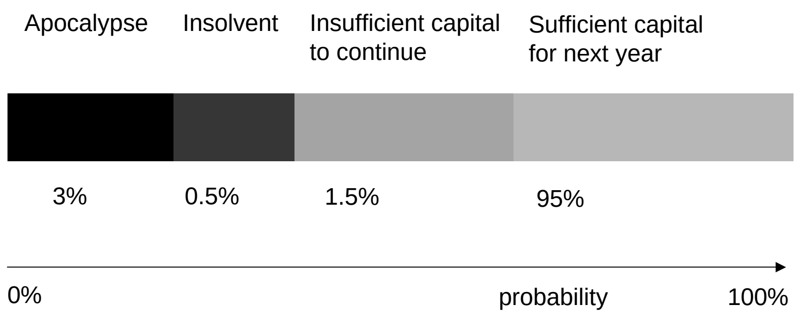
Capital ineffective

Deep pocket effect

asteroid strike
cancer cure
global warming
currency controls
flu epidemic kills 40%
employee right creep
anthrax in air con
gulf stream diversion
mass terror
nanotechbot epidemic
strikes
nuclear war
firm terrorist infiltration
AIDS – the sequel
messiah arrives
punitive WTD damages
new regulations
banking system collapse
civil disorder / insurrection
religious right – single sex offices
key person targeted
GM monsters
3 month power cut
assets frozen (WOT)
MRSA closes all hospitals
board declared unfit/improper
aliens from outer space
rogue trader / underwriter
controls violate privacy law
sharia law bans interest and insurance
customers / directors detained (WOT)
Equitable bail-out
virus / hackers destroy systems
mafia take-over
retrospective compensation
management fraud
animal rights extremists
retrospective tax
MIB for pensions
office seized for refugees
asset confiscation

Economic Capital: Who to Trust?

Scope Plan



Interpret “Capital 99.5%” as conditional on the apocalypse not happening.

Does 99.5%-ile make sense?

Conclusions

- Existing familiarity of value-at-risk gives it a head start over other approaches.
- Data and scope, but not maths, are the limiting factors for accurate capital calculations.
- If you prefer Monte Carlo, use importance sampling to cut burden by a factor of 5.
- Analytic large deviation theory is as good as 200,000 simulations – but much faster.

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