Underwriting Risk and Optimal Growth Rate for P&C Insurers

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Abstract
It is generally well established that new business generates higher loss and expense ratios and lower retention ratios than renewal business. Higher than "normal" new business growth rates generate higher means and variances of combined ratios and thus greater underwriting risks. This study investigates the optimal growth rate that balances underwriting profit requirements and the added risk associated with various rates of growth. Analytical models are derived within a classical mean-variance framework. Five models are proposed to determine the optimal growth rate for a book of business. The mean-downside-variance model is introduced for the first time in the property and casualty actuarial literature. The models can be easily expanded to incorporate non-underwriting risks, such as investment and catastrophe risks. The proposed methods are straightforward and can be effectively employed by property and casualty actuaries in support of an insurer's strategic planning process.

Key words: Growth Rate, Combined Ratio, Underwriting Risk, Downside Risk, and Mean-Variance.

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1. Introduction

Long-term growth is an important goal of property and casualty (P&C) insurer. Marketing plans, including such elements of growth as market penetration, new agency appointments, and new territory entries, are subject to overall growth control parameters of an insurer. Achieving growth generally requires the enterprise to explore new markets and/or attract new customers in existing markets. New business generally produces higher mean and variance of combined ratios. The extra underwriting expenses associated with above-average growth rates reduce profitability and the extra variability associated with growth increases the operational risk. Numerous cases have shown that rapid growth rates could cause serious financial problems to a P&C insurer, reduce long-term value to its stakeholders, or even result in a bankruptcy. According to A.M. Best, 10% of 218 P&C insolvencies from 1993-2002 were caused primarily by rapid growth.

D'Arcy and Gorvett (2004) pointed out that the new business of a P&C insurer generally produces a higher loss ratio than renewal business and the incurred loss can often be in excess of the initial premium. The loss ratios of the new business will later decline as that book of business renews. In general, a book of business would eventually be profitable after several renewals. Cohen (2001) examined over 200,000 policies within a 5-year period for an insurance company and demonstrated the favorable loss ratios associated with this aging phenomenon. Feldblum (1996) suggested that an insurance company should price the risks by the expected profitability over the expected life of the business based on the loss ratio and persistency rate of each renewal cycle. Except for pure losses, expenses on new business, such as advertising, marketing, and underwriting, are also higher than renewal business. New business generates not only higher loss and expense ratios, but also higher variance of combined ratios[1]. An aggressive growth posture obviously means a higher proportion of a book of business is made up of new business, which implies higher variance for the combined ratio, therefore greater underwriting risks. An insurance company's pace of growth depends on how much risk it can retain. Without proper risk management, an aggressive growth strategy cannot be sustained over a long period of time, and may result in significant underwriting losses. Many P&C insurers operate in a cycle of rapid growth, followed by high loss ratios, followed by rate increases to alleviate underwriting losses, followed by declines in new business because the rate is not as competitive as before. Even some major private passenger auto carriers have experienced similar ups and downs.

P&C actuaries have found that growth rates influence an insurer's loss, profit and surplus, and studied those effects using traditional actuarial and accounting methods. Muetterties (1979) presented an accounting model to calculate the necessary profit margin to keep pace with increasing premium growth. Based on rather conservative assumptions, he concluded that at least 5% before-tax underwriting margin is necessary to maintain the relationship of premium to surplus. Because the average age of exposure period may change over time as a result of growth, McClenahan (1987) examined the impact of changes in exposure growth on loss development patterns, and proposed a method to adjust the development factors. However, to date P&C actuaries have seldom studied an insurer's growth from the perspective of enterprise risk management. D'arcy and Gorvett (2004) investigated the optimal growth rate to maximize the market value of a P&C insurance company. They proposed a three-factor econometric model where an insurer's market value is hypothetically determined by its surplus, net written premium, and combined ratio. The

[1] The numerical analysis in Section 6 will show that the combined ratios of new business generate higher coefficients of variation (defined as standard deviation divided by mean) than renewal business. The author also examined the combined ratios of private passenger auto and homeowners using a P&C insurance company data, and obtained the same results.

method requires the market value data in the regression process, which may not be available for mutual P&C companies. In this study, the author examines the numerical relationship between growth rates and underwriting risks using less extensive data requirements. Analytical models are derived based on the equilibrium theory of economics within a mean-variance framework. Optimal growth rates are calculated subject to different objective functions and constraints. All the required data should be available from a P&C company's actuarial database. The models are built up on classic financial economic theories, therefore are relatively easy to understand, and can be effectively implemented by actuaries in the strategic financial planning for P&C insurance companies. The paper is organized as follows:

- Section 2 analyzes the "supply" curve of growth rates, discusses the positive correlation between growth rates and combined ratios, and introduces the concept of equilibrium new business percentage ("ENBP").
- Section 3 introduces and discusses three measurements of underwriting risks, i.e., the variance of profit, Value-at-Risk, and downside variance.
- Section 4 investigates the constraint on growth, and discusses the "demand" curve of insurer's growth rate.
- Section 5 combines the "supply" and "demand" curves discussed in Sections 2 and 4. The optimal growth rates are proposed under the following scenarios: the company wishes to (1) obtain a target combined ratio; (2) meet a target premium to surplus ratio; (3) maximize the amount of underwriting profit; (4) maximize mean-variance utility; and (5) maximize the mean-downside-variance utility.

- Section 6 shows a numerical example for a case study of private passenger auto. The numerical relationships between growth rate and underwriting profit/risk are discussed and the optimal growth rates are calculated for each of the five scenarios.
- Section 7 provides a summary of the conclusions.
- Appendix illustrates how to incorporate other risk elements, such as investment and catastrophe risks, into the model.

2. Growth Rates and Combined Ratio

Let L_r and E_r denote the loss and expense ratios for renewal business (RB), respectively. The combined ratio is $C_r = L_r + E_r$. Using the same notation convention, the loss, expense, and combined ratios for new business (NB) are denoted by L_n, E_n, and $C_n = L_n + E_n$, respectively. As noted in the introduction, L_n is generally greater than L_r, and E_n is generally greater than E_r, and therefore C_n is generally greater than C_r. Let b denote the difference between the NB and RB combined ratios, $b = C_n - C_r$. In other words, b can be considered a measure of the performance of new business relative to renewal business. Two elements make up the value b: additional costs (such as additional commissions, additional marketing expense, and adverse selection) and additional risk associated with NB levels (consisting of the greater uncertainty associated with an untested book of business). We may reasonably conclude that the faster the growth rate, the greater the b value.

Now we construct the first governing relationship:

$$b = b_0 + \alpha R + \varepsilon, \quad b_0 > 0, \ \alpha > 0 \qquad (1)$$

Where R is the overall growth rate; b_0 is the combined ratio difference between NB and RB with zero net growth (that is, the volume of NB is just sufficient to offset the non-renewal activity), αR denotes additional expenses needed to finance positive overall growth occurring at the rate R plus the cost of extra risks associated with this positive growth. In effect α measures the marginal cost of growth: the higher the α, the less cost efficient the growth. The quantity ε represents a random disturbance term and $E(\varepsilon)=0$. Let P_r and P_n be the persistency rates for renewal and new business, respectively[2]. Let A denote the proportion of the overall book of business that NB represents, then the overall combined ratio is given by C:

$$C = AC_n + (1-A)C_r = C_r + Ab. \tag{2}$$

An insurance company needs to attract more customers than those that do not renew in order for it to grow in an absolute sense, so A is an increasing function of growth rate R, that is, $\frac{dA}{dR} > 0$. By (2), $\frac{dC}{dR} = b\frac{dA}{dR} + A\alpha > 0$. Then the combined ratio C is also an increasing function of the growth rate. Suppose an insurance company writes premium 1 in the current year, and plans to grow R annually. A is from NB and 1-A is from RB. The total planned written premium in the next year is 1+R. The renewal business from the current NB is AP_n and the renewal business from the current RB is $(1-A)P_r$. The total renewal business in the following year is $AP_n + (1-A)P_r$. In the state of equilibrium, the new business percentage is constant. So the new business is $(1+R)A$.

$$AP_n + (1-A)P_r + (1+R)A = 1+R. \tag{3}$$

[2] The renewal rates are different depending on the age of the policies. To simplify the question, the author splits a book of business into NB and RB. The models can be expanded to allow multiple segmentations of the whole book by age, such as new business, first renewal, second renewal, and so on.

From (3), we can derive the equilibrium new business percentage (ENBP) required by growth rate *R*:

$$A = \frac{1 + R - P_r}{1 + R + P_n - P_r} = 1 - \frac{P_n}{1 + R + P_n - P_r}. \tag{4}$$

From (4) we have $\frac{dA}{dR} = \frac{P_n}{(1 + R + P_n - P_r)^2} > 0$, which confirms that A is an increasing function of R.

The following example illustrates the concept of ENBP. Suppose a company whose current portfolio consists of 10% new business plans to grow its total written premium by 15% annually. The annual persistency rates for new and renewal business are 80% and 90%, respectively. The renewal business in the 2nd year is 0.1*80%+0.9*90%=0.89. If the growth follows the plan, the total premium is 1.15. So the new business needs to be 1.15-0.89=0.26. The NB percentage is 0.26/1.15=22.6%. The renewal business in the 3rd year is 0.89*0.9+0.26*0.8=1.009. For 15% annual growth, the overall premium is $1.15^2 = 1.323$. So the volume of new business is 1.323-1.009=0.314. Following the same algorithm, Table 1 lists the required new business percentages for the next 5 years if the company wishes to grow its total premium by 15% annually.

Table 1: Required NB percentage needed to achieve 15% overall growth when 10% of the current book of business consists of NB

Year	RB (1)	NB (2)	RB % (3)=(1)/(5)	NB % (4)=(2)/(5)	Premium (5)=(1)+(2)
1	0.900	0.100	90.0%	10.0%	1.000
2	0.890	0.260	77.4%	22.6%	1.150
3	1.009	0.314	76.3%	23.7%	1.323
4	1.159	0.362	76.2%	23.8%	1.521
5	1.333	0.416	76.2%	23.8%	1.749

Similarly, Tables 2 and 3 report the required new business percentages if current NB percentage is 15% and 25%. By comparing tables 1-3, it is clear that ENBP is determined by the planned growth rate and persistency rates of NB and RB, and that ENBP is independent of current NB and RB composition of a book of business. However, this does not imply that the current mix of NB and RB is irrelevant with to growth plan. When current growth is slow (e.g., NB is 10% of total business), the required NB growth in the 2nd year is 0.26/0.1-1=160%. The 15% overall premium growth plan is aggressive. On the other hand, when current growth is rapid (e.g., NB is 25% of total business), the required NB growth in the 2nd year is 0.275/0.25-1=10%. The 15% growth plan is relatively easier to implement.

Table 2: Required NB percentage needed to achieve 15% overall growth when 15% of the current book of business consists of NB

Year	RB	NB	RB %	NB %	Premium
1	0.850	0.150	85.0%	15.0%	1.000
2	0.885	0.265	77.0%	23.0%	1.150
3	1.009	0.314	76.3%	23.7%	1.323
4	1.159	0.362	76.2%	23.8%	1.521
5	1.333	0.416	76.2%	23.8%	1.749

Table 3: Required NB percentage needed to achieve 15% overall growth when 25% of the current book of business consists of NB

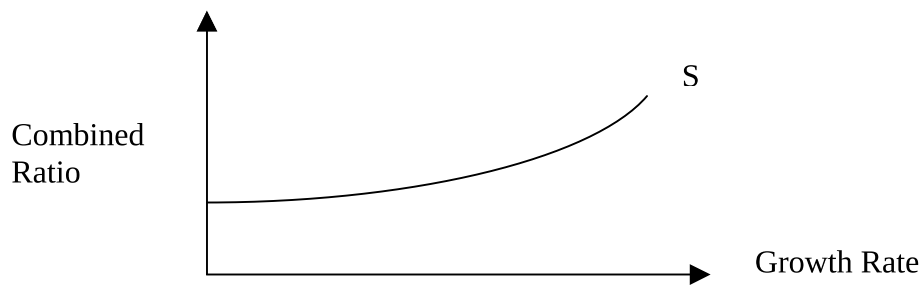
Year	RB	NB	RB %	NB %	Premium
1	0.750	0.250	75.0%	25.0%	1.000
2	0.875	0.275	76.1%	23.9%	1.150
3	1.008	0.315	76.2%	23.8%	1.323
4	1.159	0.362	76.2%	23.8%	1.521
5	1.333	0.416	76.2%	23.8%	1.749

Substitute (4) into (2), the combined ratio C is the weighted average of new and renewal business:

$$C = (1-A)C_r + AC_n = \frac{(1+R-P_r)C_n + P_n C_r}{1+R+P_n-P_r} = C_n - \frac{P_n b}{1+R+P_n-P_r}. \quad (5)$$

Equation (5) shows the positive correlation between the combined ratio and growth rate, as illustrated in Figure 1.

Figure 1: The "Supply" Curve of Growth Rate[3]



3. Growth Rate and Underwriting Risk

Let $\pi_t(R)$ be the underwriting profit at time t with growth rate R. The premium at t is $(1+R)^t$ and the profit rate is $1-C$, so the profit is $\pi_t(R)=(1+R)^t(1-C)$. Substituting the value of C from (5) yields the analytical relationship between growth rates and profit.

$$\pi_t(R)=(1+R)^t(1-C_n+\frac{P_n b}{1+R+P_n-P_r}) \qquad (6)$$

In this study, the author uses three methods to measure the underwriting risk: the variance of underwriting profit/loss, Value-at-Risk (VAR), and downside variance.

The variance reflects the magnitude of uncertainty in underwriting results, and how widely spread the values of the underwriting profit or loss are likely to be. VAR is the minimum underwriting profit over a target horizon such that there is a low, pre-specified probability that the profit will be less. VAR provides actuaries with a summary measure of underwriting risks. For instance, an insurance company might say that the VAR of its profit

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[3] The traditional supply curve shows how the cost increases with the increase in the production in an enterprise. Here the so-called "supply" curve demonstrates how the combined ratio increases with the growth rate for a P&C insurer.

ratio is -0.2 at the 99% confidence level. In other words, there is only one chance in a hundred, under normal market conditions, that a net underwriting loss greater than 20% will be realized.

Downside variance measures how widely spread the underwriting profit is likely to be below the minimal acceptable value.

$$DV = \int_{-\infty}^{T} (\pi - T)^2 dF(\pi). \qquad (7)^{4}$$

Where T is the minimal acceptable value[5] and $F(\pi)$ is the cumulative probability function of π. Variance defines the risk as how far the profit could deviate from the mean. Therefore all the variations, both desirable and undesirable, are considered as risk. Markowitz (1959) realized the drawbacks of variance as a measure of risk, as it penalizes both upside and downside movements equally. Unlike variance, downside variance only treats the unfavorable variations (e.g., the combined ratio above one) as risks. Large favorable swings will lead to a large variance, but insurers certainly do not have a problem with such favorable variation in underwriting results. Downside variance does not change with favorable deviations and appears to be a superior measure of risks. A growing number of researchers and practitioners are applying downside variance in various financial applications, especially in portfolio management. However, P&C actuaries seldom use the downside variance as a risk management tool in research and practice[6].

The following example illustrates the difference between variance and downside variance. Suppose there are three companies A, B, and C. The market could go soft or hard

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[4] When the mean of combined ratio is equal to the minimal acceptable value and it follows a symmetric distribution, the downside variance is equal to half the variance.

[5] The minimal acceptable profit is selected to be zero in this study, that is $T = 0$. In this case, only the combined ratios above one contribute to the risk calculation

[6] Berliner (1997) compared semi-variance against variance as risk measures. The semi-variance in that study is a special case of downside variance with the mean as the minimal acceptable value.

with equal probability. In case of a soft market, A, B, and C will have profits 0.1, -0.05, and -0.2. When the market is hard, A, B, and C will have profits 0.4, 0.05, and -0.2. The variances of profits for A, B, and C are 0.0225, 0.0025, and 0. Although company A has the highest variance, it has no risk at all. No matter the market is hard or soft, company A will make money. Company C has zero variance and no variability, but the risk is the highest: it will lose money under all market conditions. In this example, variance fails to reflect the true risks faced by each company. By downside variance, only net losses are included in the risk calculation, company A has no risk while company C has the highest risk (the downside variances for A, B, and C are 0, 0.00125, and 0.04, respectively).

4. The Constraints on Growth Rate

The growth of an insurance company generally is constrained by internal and external conditions, such as the availability of surplus, demands from shareholders, and market conditions. Different companies have different growth requirements. In this study, three common constraints of growth are discussed: constant growth rate, constant profit rate, and constant premium to surplus ratio. To maintain a constant growth, the demand curve is independent of the combined ratio. The company would like to grow regardless of its underwriting results. This is illustrated by the vertical demand curve in Figure 2. In case of constant profit rate, the demand curve is independent of the growth rate. The company would like to reduce its size to maintain the target profit margin. This is illustrated by the horizontal demand curve in Figure 3.

Figure 2: The "Demand" Curve for Constant Growth Rate[7]

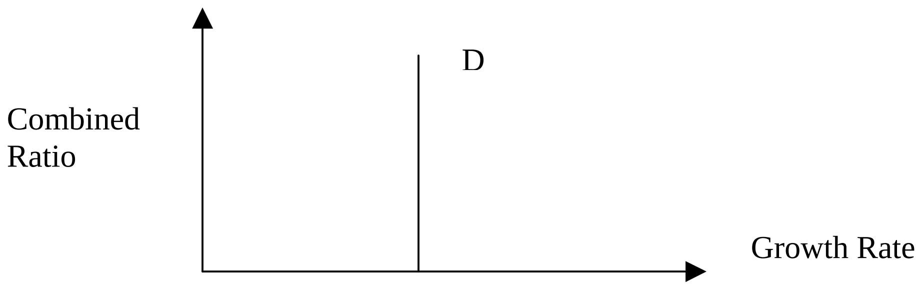

Figure 3: The "Demand" Curve for Constant Profit Rate

Past studies have shown that the availability of surplus sometimes constrains the growth of an insurance company (Hagstrom 1981, Muetterties 1979, and Davis 1979). Suppose an insurance company has a current and target premium to surplus ratio K[8]. Its current premium is 1 unit and the growth rate is R. The premium after 1 year is 1+R. Assuming no dividend to stockholders and policyholders, the surplus after 1 year is $1/K + (1+R) * (1-C)$.

To maintain the target premium to surplus ratio, $\frac{1+R}{1/K + (1+R)(1-C)}$ has to equal K. That is:

$$C = 1 - \frac{R}{K(1+R)}. \qquad (8)$$

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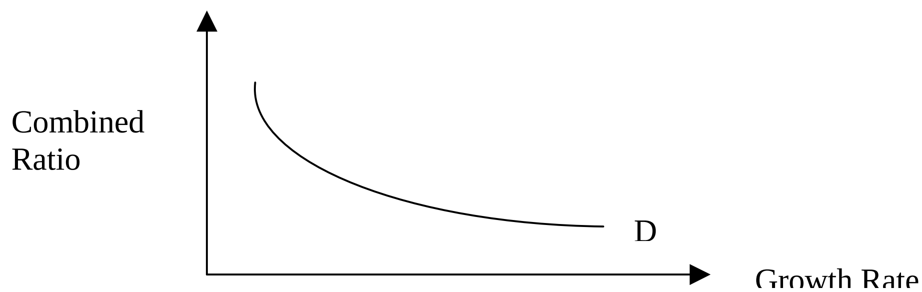
[7] The traditional demand curve shows the decreasing amount at which consumers would buy a product as its price increases. Here the so-called "demand" curve demonstrates the level at which the combined ratio decreases as the growth rate increases.

[8] In numerical analysis, we assume K=2.

By (8), the combined ratio is a decreasing function of growth rate. The implication is that "an insurance company requires a lower combined ratio for a faster growth to maintain the target premium to surplus ratio". This is illustrated by the decreasing demand curve in Figure 4.

Figure 4: The "Demand" Curve for Constant Premium to Surplus Ratio

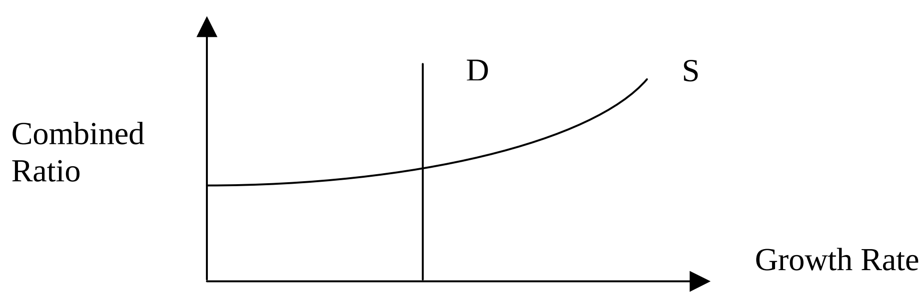


5. Optimal Growth Rate

The optimal growth rate is determined by the supply and demand curves. To meet the target growth rate R_T, the equilibrium point is shown in Figure 5. By formula, the equilibrium satisfies:

$$\begin{cases} C = C_n - \dfrac{P_n(C_n - C_r)}{1 + R + P_n - P_r} \\ R = R_T \end{cases} \qquad (9)$$

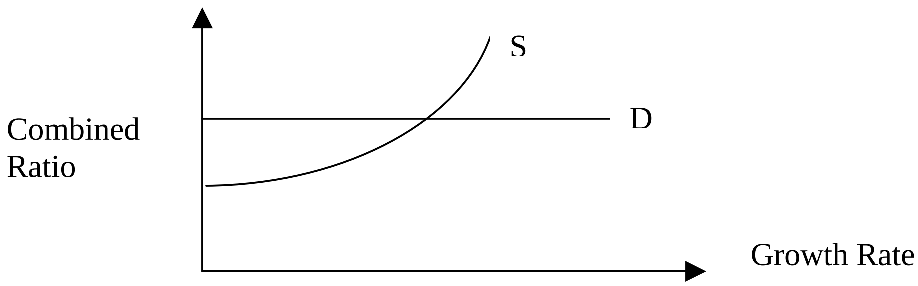
Figure 5: The Equilibrium to Meet Target Growth Rate



To meet the target profit rate τ_T, the equilibrium point is shown in Figure 6. By formula, the equilibrium satisfies:

$$\begin{cases} C = C_n - \dfrac{P_n(C_n - C_r)}{1 + R + P_n - P_r} \\ C = 1 - \tau_T \end{cases} \qquad (10)$$

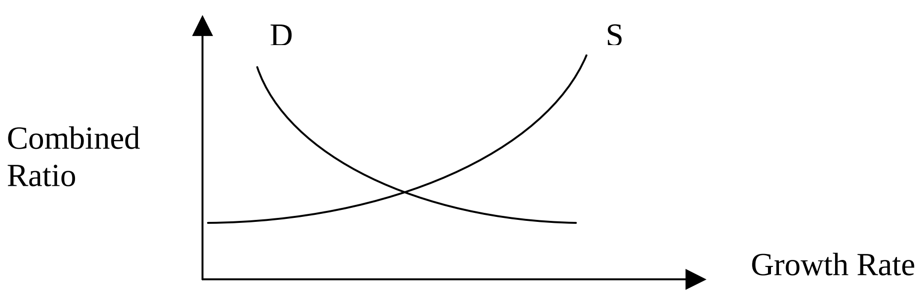
Figure 6: The Equilibrium to Meet the Target Profit Rate



To meet the target premium to surplus ratio K, the equilibrium point is shown by Figure 7. By formula, the equilibrium is:

$$\begin{cases} C = C_n - \dfrac{P_n(C_n - C_r)}{1 + R + P_n - P_r} \\ C = 1 - \dfrac{1}{K} + \dfrac{1}{K(1+R)} \end{cases} \qquad (11)$$

Figure 7: The Equilibrium to Meet the Target Premium to Surplus Ratio



An alternative way to obtain optimal growth rate is by maximizing the mean-variance utility function of an insurance company. Within mean-variance (MV) framework, the utility is positively correlated with profit and negatively correlated with risk:

$$U = Mean(\pi_t) - 0.5 * \lambda * Var(\pi_t), \qquad (12)$$

where λ is the risk aversion coefficient. When the company is risk neutral, $\lambda = 0$, maximizing utility is equivalent to maximizing profit. λ exacts a utility penalty for sustaining the risk, the greater the λ, the more risk averse. $0.5 * \lambda * Var(\pi_t)$ is the so-called risk premium: a risk-averse investor would require an additional compensation of $0.5 * \lambda * Var(\pi_t)$ to invest on a risky asset other than a risk-free asset. The author will use λ=0.2 and 0.4[9] later as examples to illustrate how the growth decision is affected by risk aversion.

Figure 8 shows the optimal growth rate to maximize the utility. U1, U2, and U3 represent the indifference curves[10] of utility, respectively. The horizontal axis is the growth rate and the vertical axis is the combined ratio. If growth rate is fixed, the lower the combined ratio, the greater the utility. If combined ratio is fixed and below 1, the higher the growth rate, the higher the utility. All the points on the same indifference curve have identical utility and all the points above the curve have lower utilities. To maintain a certain level of utility, low growth rates have to be compensated by low combined ratios so that the indifference curves are going downward as growth rates become smaller. U1 is above U2 and U3 so that it implies the greater combined ratios for the same growth rates. Therefore, U1 represents lower utility than U2 and U3. S is the supply curve of growth rate. The

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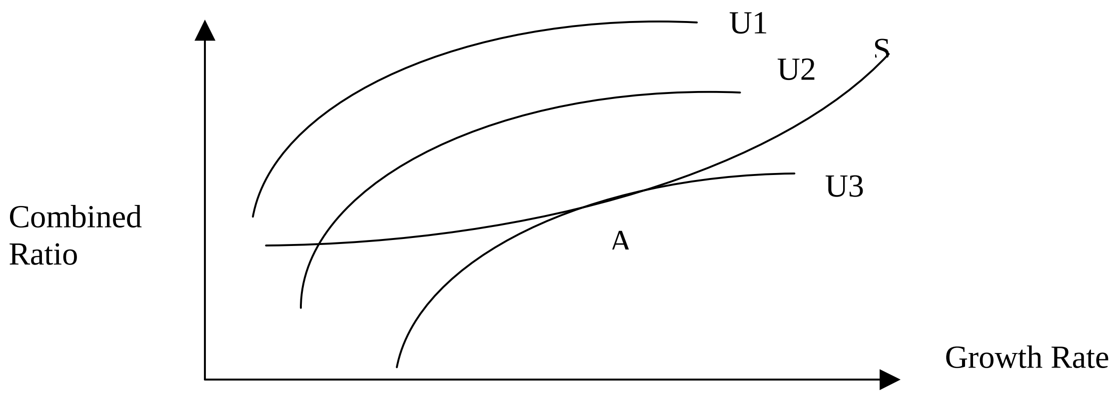
[9] By the author's experience, when $\lambda > 0.5$, the utility penalty on risk bearing is too severe. $\lambda = 0.2$ implies the risk load on premium is 10% of variance and $\lambda = 0.4$ implies the risk load is 20%.

[10] Indifference curve is defined as the line on which all the points have the same utility.

gradient point A between the supply curve and indifference curve represents the optimal growth rate. All other points on the supply curve are above the indifference curve U3, and therefore have lower utility values than A.

Figure 8: The Equilibrium to Maximize the Utility



Let $\mu(R)$ and $V(R)$ denote the mean and variance of the combined ratio C associated with growth rate R, which can be derived from equation (5). In the context of this study, the expected profit at time t is $mean(\pi_t) = (1+R)^t(1-\mu(R))$ and the variance of profit is $Var(\pi_t) = (1+R)^{2t}V(R)$. The optimal growth rate is selected to maximize:

$$\underset{R}{Max}\ \ U(\pi_t) = (1+R)^t(1-\mu(R)) - 0.5\lambda(1+R)^{2t}V(R), \qquad (13)$$

The maximization problem can be solved by the first order condition of (13):

$$\frac{dU}{dR} = -(1+R)^t\frac{d\mu}{dR} + t(1+R)^{t-1}(1-\mu(R)) + 0.5\lambda(1+R)^{2t}\frac{dV}{dR} + \lambda t(1+R)^{2t-1}V(R) = 0. \qquad (14)$$

Using downside risk instead of variance, the MV utility function becomes mean downside-variance (MDV) utility, which is becoming more popular in financial applications.

$$U = Mean(\pi_t) - \lambda * DV(\pi_t). \qquad (15)$$

The rationale of (15) is: if π_t is symmetrically distributed and the minimal accepted return is $mean(\pi_t)$, then $DV(\pi_t) = 0.5 * Var(\pi_t)$. Let $DV(R)$ denote the downside variance of the combined ratio C associated with growth rate R, the optimal growth rate can be obtained by maximizing:

$$\underset{R}{Max} \quad U(\pi_t) = (1+R)^t (1-\mu(R)) - \lambda(1+R)^{2t} DV(R). \qquad (16)$$

6. Numerical Analysis

A numerical analysis is given in the example of private passenger auto insurance. The underwriting profit and risk are evaluated for the 5th year to reflect the general length of strategic planning. The combined ratios of NB and RB are assumed to be normally distributed:

$$\begin{aligned} &C_r \sim normal(\mu_r, \mu_r^2 v_r^2) \\ &C_n \sim normal(\mu_n, \mu_n^2 v_n^2). \\ &Corr(C_r, C_n) = \rho \end{aligned}$$

Where μ_r and μ_n are the means of combined ratio for renewal and new business, respectively, and $\mu_n - \mu_r = E(b) = b_0 + \alpha R$. μ_r represents the expected RB performance and $E(b)$ represents the expected NB performance relative to RB. v_r and v_n are the coefficients of variation for the combined ratios for renewal and new business, respectively; ρ is the correlation coefficient between C_r and C_n. Mildenhall (1999) and McCullagh and Nelder (1989) contended that the variance of insurance data is usually positively correlated to the mean. So, the constant coefficient of variation is a more realistic assumption than the constant variance. Because C_r and C_n are normally distributed, the overall combined ratio is normally distributed:

$$C \sim normal(A\mu_n + (1-A)\mu_r, A^2\mu_n^2 v_n^2 + (1-A)^2 \mu_r^2 v_r^2 + 2A(1-A)\rho\mu_n v_n \mu_r v_r)$$

Where A is the ENBP and $A = 1 - \frac{P_n}{1 + R + P_n - P_r}$.

The key parameters are estimated using the private passenger auto data of an insurance company. The annual persistency rates of renewal and new business are calculated and rounded to the closest 0.05, $P_r = 85\%$ and $P_n = 75\%$. b_0 and α are estimated using linear regression. The response variable is the combined ratio difference between NB and RB, and the explanation variable is the annual growth rate of direct written premium, $b_0 = 0.1243$ and $\alpha = 0.4676$. μ_r is selected as 0.92, which is the median of RB combined ratios of the subject book of business. The values of v_r and v_n are calculated and rounded to the nearest 0.05 using the same dataset, $v_r = 0.15$, $v_n = 0.20$. ρ is set to be 0.7[11].

Table 4 reports the size of business, ENBP, expected combined ratio, and expected profit with growth rates from –10% to 20% by 2.5% increments. Table 5 shows the risk measurements (i.e. the variance of profit, VAR at 95% confidence level, and the downside variance associated with those growth rates. Figure 9 displays visually the relationship between profit and growth; while Figures 10-12 display how the three risks respond to growth.

From Table 4, the ENBP increases and combined ratio decreases with growth rate, this is consistent with the theory in Section 2. The profit first increases with growth rate and then drops. Profit is jointly determined by the size of the business and the profit rate. Although the profit rate declines with growth, growth dominates in determining profit at the lower growth rates. When growth rate becomes larger, the combined ratio will continue to increase. After the growth rate reaches a certain level, the profit rate dominates growth, and

[11] The author calculated the correlations between RB and NB for the subject book of business, which varies from 0.4 to 0.9.

the profit decreases with growth. From Table 5, all three measurements of risk are the increasing functions of the growth rate. The implication is that greater growth rates also bring greater risks to a P&C insurer. For example, comparing the change in risk as the growth rates move from 0% to 20%, the variance increases from 0.0194 to 0.1427; the downside variance increases from 0.0047 to 0.0637. Furthermore, the VARs of profit are all negative, which implies that there is a 5% chance of an underwriting loss even if the business is shrinking. This is determined by the risky nature of insurance.

From Figures 9-12 and Table 5, all the risks increase with growth rate. Both variance and downside variance accelerate with growth, as shown by the convex increasing curves. For example, the variance and downside variance with –10% growth are 0.0064 and 0.0012, respectively. With 20% growth rate, these two values increase to 0.1427 and 0.0637, respectively. The increases are 53.08 (0.0637/0.0012) and 22.30 (0.1427/0.0064) for downside-variance and variance. The downside variance increases more rapidly than variance. This is because all the observations are used to calculate variance, but only the observations with combined ratio above one are applied to calculate the downside variance. The greater the growth rate, the more likely the combined ratio exceeds one.

Table 4: Underwriting Profits with Growth Rates

Growth	Size of Business	Expected Combined Ratio	Expected Profit
-10.0%	0.59	0.925	0.0444
-7.5%	0.68	0.928	0.0487
-5.0%	0.77	0.932	0.0527
-2.5%	0.88	0.936	0.0563
0.0%	1.00	0.941	0.0593
2.5%	1.13	0.946	0.0614
5.0%	1.28	0.951	0.0624
7.5%	1.44	0.957	0.0621
10.0%	1.61	0.963	0.0600

12.5%	1.80	0.969	0.0558
15.0%	2.01	0.976	0.0492
17.5%	2.24	0.982	0.0396
20.0%	2.49	0.989	0.0266

Table 5: Underwriting Risks measured by profit

Growth	Variance	Value at Risk	Downside Variance
-10.0%	0.0064	-0.087	0.0012
-7.5%	0.0085	-0.103	0.0017
-5.0%	0.0112	-0.122	0.0024
-2.5%	0.0148	-0.144	0.0033
0.0%	0.0194	-0.170	0.0047
2.5%	0.0252	-0.200	0.0065
5.0%	0.0327	-0.235	0.0091
7.5%	0.0423	-0.276	0.0127
10.0%	0.0544	-0.324	0.0177
12.5%	0.0696	-0.378	0.0245
15.0%	0.0888	-0.441	0.0339
17.5%	0.1128	-0.513	0.0465
20.0%	0.1427	-0.595	0.0637

Figure 9: ENBP by Growth

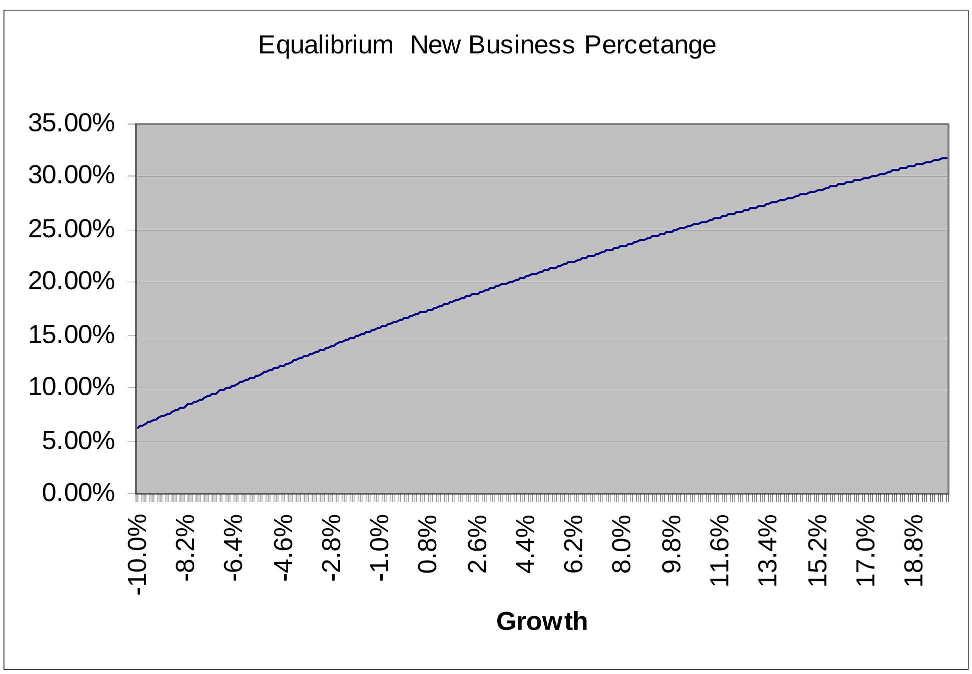


Figure 10: Variance of Profits by Growth Rates

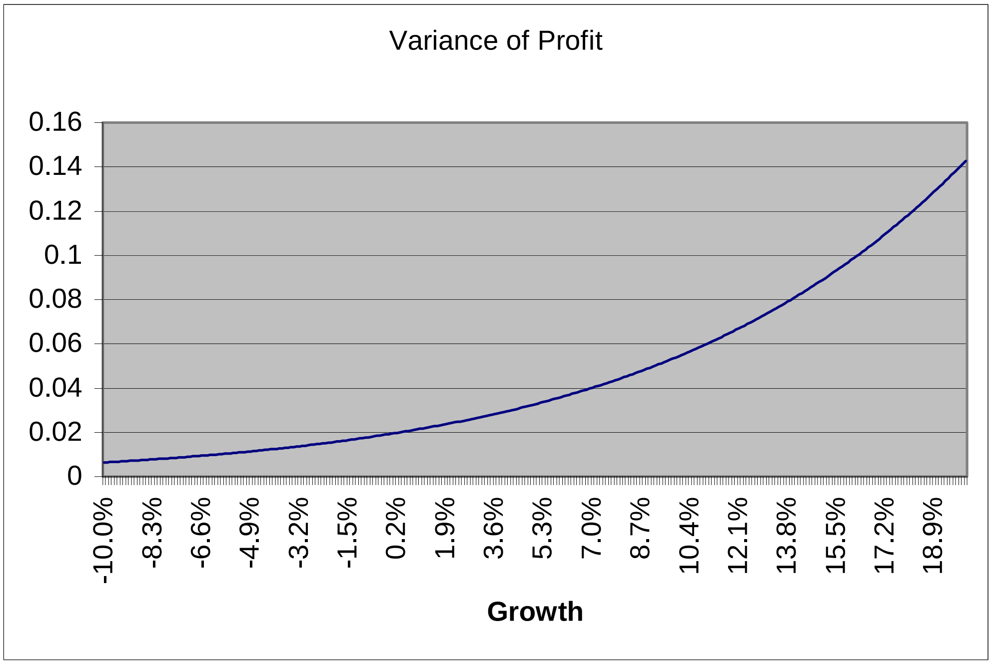


Figure 11: 95% VAR of Profit by Growth Rate

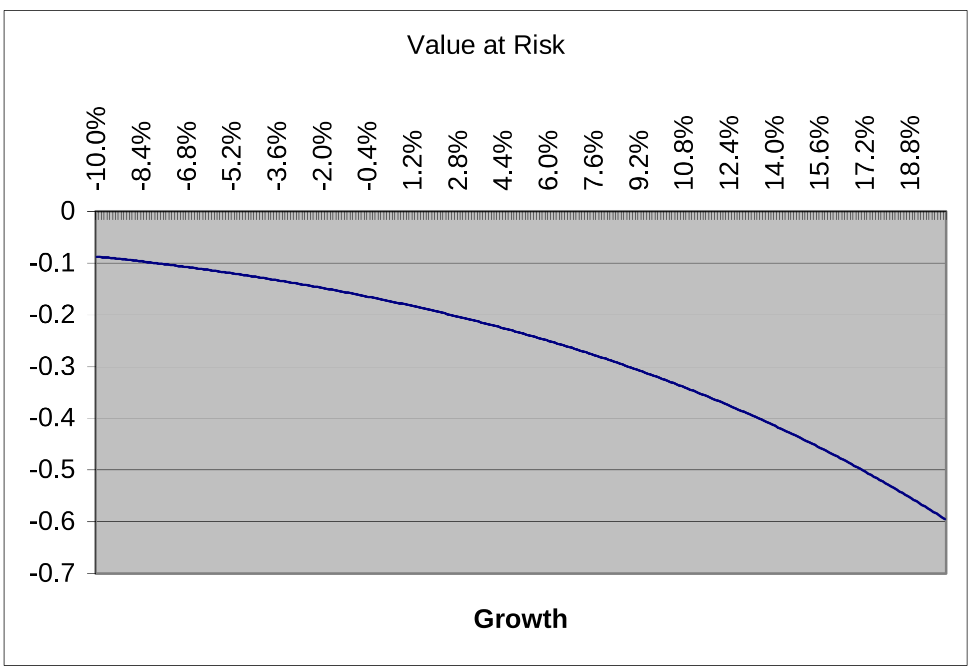
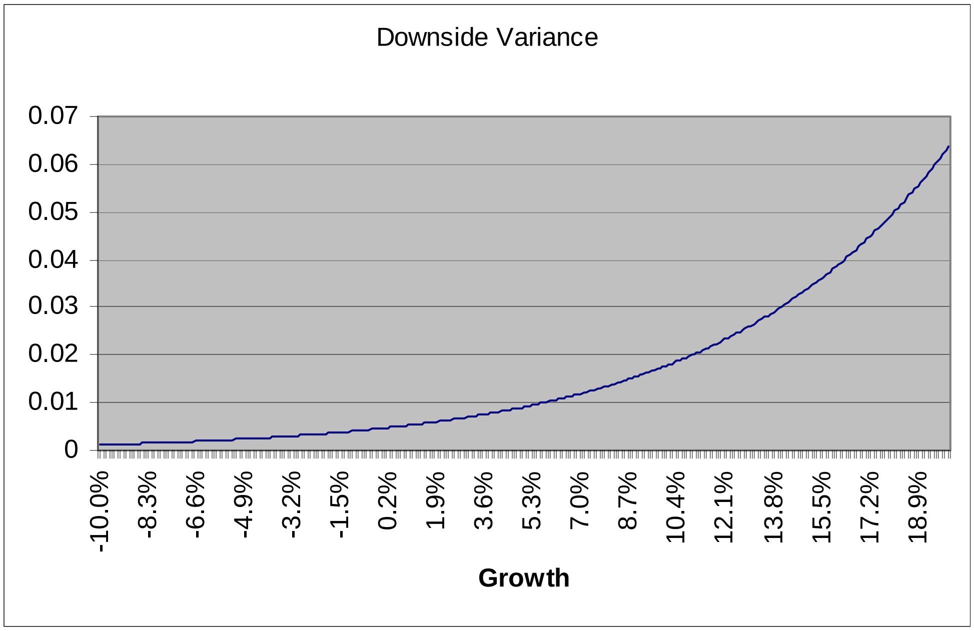


Figure 12: Downside Variance of profits by growth Rates



The optimal growth rates are solved numerically[12] and reported in Table 6. Eight growth scenarios are discussed:

- Scenario 1: to meet a target growth rate of 5%;
- Scenario 2: to meet a target combined ratio of 0.95 (or an underwriting profit of 5%);
- Scenario 3: to meet a target premium to surplus ratio of 2.
- Scenario 4: to maximize profit in the 5th year.
- Scenario 5: to maximize the MV utility in the 5th year with $\lambda = 0.2$.
- Scenario 6: to maximize the MV utility in the 5th year with $\lambda = 0.4$.
- Scenario 7: to maximize the MDV utility in the 5th year with $\lambda = 0.2$.
- Scenario 8: to maximize the MDV utility in the 5th year with $\lambda = 0.4$.

[12] It is difficult to derive the analytical solutions for utility-maximizing models, especially for the MDV models. Golden section search method is used to solve the utility maximization problem numerically.

From Table 6, the growth rate needs to be 4.5% to meet the target 5% underwriting profit rate. To maintain the premium to surplus ratio at 2, the growth rate needs to be 8.8%. If the company grows faster than 8.8%, its premium would increase faster than surplus, and the premium to surplus ratio will be larger than 2. A 5.7% growth rate will maximize the underwriting profit. In the utility maximization methods, the risk associated with more rapid growth exacts a penalty on the expected return. The larger the risk aversion coefficient λ, the higher the risk penalty is given to the utility. This explains why a larger risk aversion requires a lower growth rate. In practice, actuaries should select λ based on their actuarial judgments for an insurer by its financial strength, risk perception attitude, long-term development strategy, and position in the market. Table 5 shows that when the book is profitable, the variance is more than two times the downside variance so that the MV utility exacts a larger risk penalty than the MDV utility. Therefore the MDV optimal growth rates are higher than the MV rates.

Table 6: The Optimal Growth Rates

Case	Growth Rate	Combined Ratio	Profit
1	5.0%	0.9511	0.0624
2	4.5%	0.9500	0.0623
3	8.8%	0.9599	0.0612
4	5.7%	0.9527	0.0625
5	4.2%	0.9493	0.0622
6	3.0%	0.9468	0.0617
7	4.6%	0.9502	0.0623
8	3.7%	0.9483	0.0620

7. Conclusions

One of the critical decisions faced by a P&C insurance company is to determine the optimal level of premium growth. Marketing plans, such as market penetration, new agency appointments, and new territory entries, are elements of a growth plan. New business generally has higher loss and expense ratios and greater uncertainty in loss performance than renewal business. More rapid growth requires higher percentages of new business; therefore leads to higher combined ratios and greater operational risk. An insurance company's pace of growth relies on how much risk it can retain.

The author introduces a concept of ENBP and shows how to calculate the amount of new business a company needs to write in order to grow at a steady rate. Based on the ENBP, the author derives the analytical and numerical relationships between growth rates and combined ratios, and illustrates the shape of supply curve of growth rates.

Three methods are applied to measure the underwriting risk using growth rate as the explanation variable. The traditional variance makes use of all the observations, both favorable and unfavorable, to calculate the risk. VAR and downside variance define risk as a result of only the undesirable deviations. The three methods examine risk from different perspectives. The variance reflects how widely spread the underwriting results are likely to be. VAR is the minimum profit over a target horizon such that there is 5% chance that the actual profit will be less. Downside variance, measures how widely spread the net loss outcomes could be. Using a numerical example of private passenger auto, the author demonstrates how these three risk measurements increase with growth rate. The downside variance is much more sensitive to growth rates than the standard variance.

Five methods[13] are proposed to determine the optimal growth rate for insurance companies: to meet a target combined-ratio, to meet a target premium to surplus ratio, to maximize the profit, and to maximize MV and MDV utility functions. The target combined ratio method is simple and easy to apply, but the pre-determined target ratio is usually provided by management and may not be justified by either the theory or the history of the

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[13] To meet the target growth rate, the optimal rate is pre-specified. It is not included as a method to obtain optimal growth rate.

company. The profit maximization method does not take risk into consideration so that it yields optimal growth rates that are higher than those produced by the utility maximization methods. Premium to surplus ratio is a restrictive constraint for many companies, so the proposed method is realistic and practical. Utility maximization methods are popularly used by financial economists, but not with P&C actuaries. Compared to other models, the utility maximization requires an extra parameter λ -- risk aversion coefficient. This provides the actuaries with greater flexibility in calculating optimal growth rates: the higher the risk aversion, the smaller the optimal growth rate. However, the selection of λ for a company needs to be justified by its financial strength, the risk attitude of management, the long-term development strategy, and the marketing environment.

This study focuses on the underwriting risk. Other types of risks, such as investment and catastrophe risks can be incorporated into the model[14]. The author also simplifies the overall business into renew and new books. As D'arcy and Gorvett (2004) discussed, the combined ratio would reduce with each renewal cycle. This suggests the overall business could be divided into multiple components: new business, 1st renewal, 2nd renewal, and so on. Aggregating all the renewal business into one group may result in small errors.

[14] Adding other risks may make the numerical solution much complicated. The appendix shows an example of incorporating the investment and catastrophe risks into the model.

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Appendix: An Example of Incorporating More Risk Elements

In the paper, the author only considers two risks (NB and RB underwriting risks) in the model. It is relatively straightforward to extend the model to include other types of risks. Let r_i be the loss ratio for the ith risk. Then the overall combined ratio is $C+\sum r_i$. The profit during the time period *t* is $(1+R)^t(1-C-\sum r_i)$. As an illustration, one could include investment risk (r_1) and catastrophe risk (r_2) in the model. Assume that the insurer only invests in riskless assets (i.e. T-Bond) with an annual return rate of 4%, and the fund-generating coefficient is 0.8, and the tax on investment income is 0.35%, so $mean(r_1)=-0.8*0.65*4\%=-2.08\%$ and $\mathrm{var}(r_1)=0$. Also assume that catastrophe risk is independent of underwriting plans, and $mean(r_2)=1\%$ and its coefficient of variation is 10. By adding investment and catastrophe risks into the model, the profit maximizing growth rate is 8.5%, compared to 5.7% before. This is because the expected investment income is larger than expected catastrophe losses. The optimal growth rates are 6.2% and 4.4% for the mean-variance utility functions with $\lambda=0.2$ and $\lambda=0.4$, respectively. The corresponding rates for mean-downside-variance utilities are 6.8% and 5.4%[15]. The gaps between profit-maximizing growth rate and utility-maximizing rates increase because the risk penalties in the model are higher after introducing catastrophe risk.

[15] If the growth is driven by entering high catastrophic geographic regions, the catastrophe risks is positively correlated with growth targets. The utility-maximizing growth rates would be lower.