Value Creation for Insurers

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Abstract

In this paper we analyze the value creation for an insurance company. We concentrate only on the underwriting risk. We use a multivariate normal random vector in order to modelize the underwriting risk of the insurer. Our model accounts for correlations between risks and between lines of business. We compute return on risk adjusted capital (RORAC) and economic value added (EVA) for the whole conglomerate as well as for the lines of business. When there are negative correlations, we show that it may be justified to write business with negative margins. We conclude that it is dangerous to take investment or disinvestment decisions based on local figures. Only the EVA or RORAC for the whole conglomerate is relevant. We also analyse the effect of stop-loss reinsurance.

Keywords

Required solvency level, Tail Value at Risk, Risk adjusted capital, Diversification benefit, Value creation, Stochastic dependencies, Multivariate normal model.

Value Creation for Insurers

1 Introduction

In this paper we set up a simple model for the underwriting risk of an insurance company. The model accounts for dependencies within the written risks. We show how to compute the value creation for the shareholders and in particular we stress on the fact that local analyses (line of business per line of business) may lead to wrong conclusions. As an example we will see that it may be justified to write business with negative margin as soon as it lends capital to other lines of business. We will also observe that not all stochastic dependencies are allowed within a model.

Policyholders buy insurance covers in order to avoid disastrous financial consequences when an unwanted event happens. Obviously they do not want to exchange insurance risks (e.g. fire, liability, death, ...) for credit risk. They indeed want to make sure that the insurer will be able to compensate them even if there is a strong deviation of the aggregate loss compared to the premiums charged. This is why insurers must hold economic capital. This capital is provided by investors that will require a certain rate of return (cost of capital). This cost of capital is a function of the class of assets the insurer invests in and it is also a function of some frictional costs (double taxation costs, agency costs, financial distress costs) (see Hancock et al. (2001) for more details).

As explained in Hancock et al. (2001) underwriting and finance should be analysed separately. In the present paper we will only concentrate on underwriting risk. We do not work with long-tailed lines of business such that there is no financial advantage from the inversion of the business cycle. Furthermore we assume that the capital earns the expected base cost of capital : we do not account for the possibility of beating the financial market.
The rest of the paper is organized as follows. Section 2 presents the general framework for value creation. In section 3 we present our multivariate normal model. Section 4 deals with the matrix theory results that are necessary to make sure that our model is correctly defined. Numerical applications are provided in section 5. Possible extensions to the multivariate normal model are presented in section 6. Section 7 concludes.

2 Measures for Value Creation

The required solvency level (RSL) of an insurance company is the amount that the insurer has to hold in order to be able to compensate the policyholders with a given level of security. The RSL should be determined by the regulator. If we substract the amount of money that is borrowed from the policyholders, i.e. the premiums, we obtain the capital, or risk adjusted capital, provided by the investors.

A very first candidate for the RSL is the Value at Risk of level ϵ :

$$VaR_\epsilon(S) = \inf\{x \in \mathbb{R} | F_S(x) \geq \epsilon\}\,,\; 0 < \epsilon < 1.$$

However the VaR does not tell how bad is bad. It does not give an idea about the severity of the default. On top of this the VaR is not a subadditive risk measure.

These are the reasons why another measure of risk has been introduced, namely the Tail Value at Risk, which is a subadditive measure, i.e. respecting

$$\rho(S_1 + S_2) \leq \rho(S_1) + \rho(S_2)$$

where ρ denotes the chosen risk measure.

The Tail Value at Risk at level ϵ is the average of the quantiles above the Value at Risk at the level ϵ :

$$TVaR_\epsilon(S) = \frac{1}{1-\epsilon}\int_\epsilon^1 VaR_q(S)dq\,,\; 0 < \epsilon < 1.$$

For continuous random variables, we have that the Tail Value at Risk is equal to the Conditional Tail Expectation :

$$CTE_\epsilon(S) = \mathbb{E}[S | S > VaR_\epsilon(S)]\,,\; 0 < \epsilon < 1.$$

For continuous random variables, the conditional tail expectation represents the expectation of the top $(1-\epsilon)\%$ losses.

For normally distributed risks with mean μ and standard deviation σ ($S \sim Nor(\mu, \sigma)$), it is well known that

$$\begin{aligned} VaR_\epsilon(S) &= \mu + \sigma\Phi^{-1}(\epsilon) \\ CTE_\epsilon(S) = TVaR_\epsilon(S) &= \mu + \sigma\frac{\phi(\Phi^{-1}(\epsilon))}{1-\epsilon} \end{aligned}$$

where ϕ (resp. Φ) denotes the density function (resp. the cumulative distribution function) of a standard normal random variable ($Nor(0,1)$).

Artzner et al. (1999) make some recommendations about the set of axioms that should be satisfied by risk measures. The Tail Value at Risk is a coherent risk measure in the sense of Artzner et al. (1999). As from now we will compute the Tail Value at Risk in order to get our required solvency level.

The risk adjusted capital is the amount of money provided by investors. It is equal to the RSL minus the premiums that are borrowed to policyholders :

$$\begin{aligned} RAC &= RSL - P \\ &= TVaR_\epsilon(S) - P. \end{aligned}$$

The EVA (economic value added) is given by

$$EVA = Margin - kRAC$$

where k is the cost of capital demanded by the shareholders. We may also compute the RORAC ((expected) return on risk adjusted capital):

$$RORAC = \frac{Margin}{RAC}.$$

Our aim will be to maximize the EVA under a capital constraint. Moreover we will also look at the EVA per line of business. Therefore we need to allocate the full capital between the lines of business. This will be done using the Tail Value at Risk allocation method :

$$RAC_i = \mathbb{E}[S_i|S > VaR_\epsilon(S)] - P_i$$

where S_i denotes the aggregate claims for LOB i and P_i denotes the premium for LOB i.

This allocation method is additive :

$$RAC = RAC_1 + RAC_2 + RAC_3.$$

Moreover this allocation respects the no undercut axiom (see Denault (2001) for more details). Indeed using this allocation method together with a subadditive risk measure ensures that coalitions will not be favoured. For example lines of business 1 and 2 will never have an incentive to leave the conglomerate due to the fact that they would consume less capital on a stand alone basis than within the full conglomerate. This no undercut axiom is not verified for a range of classical allocation methods such as the relative allocation :

$$RAC_i = RAC\frac{K_i}{K_1 + K_2 + K_3}$$

where K_i is the stand alone RAC of line of business i :

$$K_i = \mathbb{E}[S_i|S > VaR_\epsilon(S)] - P_i.$$

When the risks are normally distributed, we deduce the following formula from Panjer (2001):

$$RAC_i = \mu_{S_i} + \sigma_{S_i}\sigma_S\rho_{S_i,S}\frac{\phi(\Phi^{-1}(\epsilon))}{1-\epsilon} - Margin_i$$

where $\rho_{S_i,S}$ denotes the correlation between S_i and S.

It then becomes easy to analyse how the value is created within the business lines :

$$EVA_i = \text{Margin}_i - kRAC_i.$$

One may also be tempted to compute the RORAC per line of business :

$$RORAC_i = \frac{\text{Margin}_i}{RAC_i}.$$

We will see that this measure may be extremely misleading.

3 Multivariate Normal Model

We now move to an insurance company writing three kinds of risks. We assume that these risks are normally distributed for the sake of simplicity. We recognize dependencies between these risks. We assume to have an amount of capital at our disposal. Our aim will be to maximize the value creation for the shareholders. We only look at technical premiums, i.e. pure premiums plus capital charges. We disregard the management of administrative and acquisition expenses.

Assume the following types of normally distributed risks :

- n_1 risks of type 1 : $X_1, \ldots, X_{n_1}$
- n_2 risks of type 2 : $X_{n_1+1}, \ldots, X_{n_1+n_2}$
- n_3 risks of type 3 : $X_{n_1+n_2+1}, \ldots, X_{n_1+n_2+n_3}$.

Assume the following "identically distributed" assumptions

$$\begin{aligned}
\mu_{X_j} &= \mu_1 \, , \, j = 1, \ldots, n_1 \\
\mu_{X_j} &= \mu_2 \, , \, j = n_1+1, \ldots, n_1+n_2 \\
\mu_{X_j} &= \mu_3 \, , \, j = n_1+n_2+1, \ldots, n_1+n_2+n_3 \\
\sigma_{X_j} &= \sigma_1 \, , \, j = 1, \ldots, n_1 \\
\sigma_{X_j} &= \sigma_2 \, , \, j = n_1+1, \ldots, n_1+n_2 \\
\sigma_{X_j} &= \sigma_3 \, , \, j = n_1+n_2+1, \ldots, n_1+n_2+n_3 \\
\rho_{X_i,X_j} &= \rho_{11} \, , \, i = 1, \ldots, n_1, j = i+1, \ldots, n_1 \\
\rho_{X_i,X_j} &= \rho_{22} \, , \, i = n_1+1, \ldots, n_1+n_2, j = i+1, \ldots, n_1+n_2 \\
\rho_{X_i,X_j} &= \rho_{33} \, , \, i = n_1+n_2+1, \ldots, n_1+n_2+n_3, j = i+1, \ldots, n_1+n_2+n_3 \\
\rho_{X_i,X_j} &= \rho_{12} \, , \, i = 1, \ldots, n_1, j = n_1+1, \ldots, n_1+n_2 \\
\rho_{X_i,X_j} &= \rho_{13} \, , \, i = 1, \ldots, n_1, j = n_1+n_2+1, \ldots, n_1+n_2+n_3 \\
\rho_{X_i,X_j} &= \rho_{23} \, , \, i = n_1+1, \ldots, n_1+n_2, j = n_1+n_2+1, \ldots, n_1+n_2+n_3 \\
P_j &= (1+\alpha_j) n_j \mu_j.
\end{aligned}$$

P_j denotes the total premium charged for policies of the line of business (LOB) j. The latter equation means that the average loading charged on policies of LOB j is α_j.

The variance-covariance matrix $\mathbf{\Sigma}$ is the $(n_1+n_2+n_3) \times (n_1+n_2+n_3)$ matrix with elements

$$\begin{aligned}
\mathbf{\Sigma}(j,j) &= \sigma^2_{X_j} \, , \, 1 \leq j \leq n_1+n_2+n_3, \\
\mathbf{\Sigma}(i,j) &= \rho_{X_i,X_j} \sigma_{X_i} \sigma_{X_j} \, , \, 1 \leq i \neq j \leq n_1+n_2+n_3.
\end{aligned}$$

In order to show the particular structure of the variance-covariance matrix in our example, we draw it for $n_1 = n_2 = n_3 = 3$:

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho_{11}\sigma_1^2 & \rho_{11}\sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 \\ \rho_{11}\sigma_1^2 & \sigma_1^2 & \rho_{11}\sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 \\ \rho_{11}\sigma_1^2 & \rho_{11}\sigma_1^2 & \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 \\ \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \rho_{22}\sigma_2^2 & \rho_{22}\sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 \\ \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{22}\sigma_2^2 & \sigma_2^2 & \rho_{22}\sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 \\ \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \rho_{22}\sigma_2^2 & \rho_{22}\sigma_2^2 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 \\ \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \sigma_3^2 & \rho_{33}\sigma_3^2 & \rho_{33}\sigma_3^2 \\ \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{33}\sigma_3^2 & \sigma_3^2 & \rho_{33}\sigma_3^2 \\ \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \rho_{33}\sigma_3^2 & \rho_{33}\sigma_3^2 & \sigma_3^2 \end{pmatrix}.$$

4 Sufficient Conditions for Positive Semidefiniteness of $\mathbf{\Sigma}$

Because $\mathbf{\Sigma}$ is a variance-covariance matrix, it is important to make sure that $\mathbf{\Sigma}$ is positive semidefinite, i.e. that

$$\boldsymbol{a}^t \cdot \mathbf{\Sigma} \cdot \boldsymbol{a} \geq 0$$

for all nonzero $(n_1 + n_2 + n_3) \times 1$ vectors $\boldsymbol{a}$.

A necessary and sufficient condition for positive semidefiniteness is that all the eigenvalues of $\mathbf{\Sigma}$ are nonnegative. This is the condition we will verify. The reader is referred to e.g. Horn and Johnson (1985) and Golub and Van Loan (1996) for more details on matrix analysis. All matrix theory results used in this paper may be found in appendix A.

The particular structure of our variance-covariance matrix may be summarized with the following notation :

$$\mathbf{\Sigma} = \begin{pmatrix} \boldsymbol{D}_{11} & \boldsymbol{D}_{12} & \boldsymbol{D}_{13} \\ \boldsymbol{D}_{21} & \boldsymbol{D}_{22} & \boldsymbol{D}_{23} \\ \boldsymbol{D}_{31} & \boldsymbol{D}_{32} & \boldsymbol{D}_{33} \end{pmatrix}$$

where $\boldsymbol{D}_{pp}$ is a $n_p \times n_p$ matrix such that

$$\begin{aligned} \boldsymbol{D}_{pp}(i,i) &= \sigma_p^2\,,\ 1 \leq i \leq n_p, \\ \boldsymbol{D}_{pp}(i,j) &= \rho_{pp}\sigma_p^2\,,\ 1 \leq i \neq j \leq n_p \end{aligned}$$

and $\boldsymbol{D}_{pq}$ is a $n_p \times n_q$ matrix such that

$$\boldsymbol{D}_{pq}(i,j) = \rho_{pq}\sigma_p\sigma_q\,,\ 1 \leq i \leq n_p\,,\ 1 \leq j \leq n_q.$$

Let us denote by $\boldsymbol{S}$ the diagonal matrix with diagonal elements $(\underbrace{\sigma_1, \ldots, \sigma_1}_{n_1 \text{ times}}, \underbrace{\sigma_2, \ldots, \sigma_2}_{n_2 \text{ times}}, \underbrace{\sigma_3, \ldots, \sigma_3}_{n_3 \text{ times}})$.

Let $\boldsymbol{R}$ be

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} & \boldsymbol{R}_{13} \\ \boldsymbol{R}_{21} & \boldsymbol{R}_{22} & \boldsymbol{R}_{23} \\ \boldsymbol{R}_{31} & \boldsymbol{R}_{32} & \boldsymbol{R}_{33} \end{pmatrix}$$

where $\boldsymbol{R}_{pp}$ is a $n_p \times n_p$ matrix such that

$$\begin{aligned} \boldsymbol{R}_{pp}(i,i) &= 1\,,\, 1 \le i \le n_p, \\ \boldsymbol{R}_{pp}(i,j) &= \rho_{pp}\,,\, 1 \le i \ne j \le n_p \end{aligned}$$

and $\boldsymbol{R}_{pq}$ is a $n_p \times n_q$ matrix such that

$$\boldsymbol{R}_{pq}(i,j) = \rho_{pq}\,,\, 1 \le i \le n_p\,,\, 1 \le j \le n_q.$$

We have

$$\boldsymbol{\Sigma} = \boldsymbol{S} \cdot \boldsymbol{R} \cdot \boldsymbol{S}^t.$$

Because $\boldsymbol{S}$ is nonsingular (we indeed exclude degenerate cases where $\sigma = 0$), $\boldsymbol{\Sigma}$ is congruent to $\boldsymbol{R}$. This implies that if the eigenvalues of $\boldsymbol{R}$ are nonnegative, then the eigenvalues of $\boldsymbol{\Sigma}$ are nonnegative as well. Indeed symmetric and congruent matrices have the same inertia according to Sylvester's law of inertia. The problem is thus reduced to the analysis of the sign of the eigenvalues of $\boldsymbol{R}$. The values of σ_i do not play a role in our problem.

Let $\boldsymbol{U}$ be

$$\boldsymbol{U} = \begin{pmatrix} \boldsymbol{U}_1 & 0 & 0 \\ 0 & \boldsymbol{U}_2 & 0 \\ 0 & 0 & \boldsymbol{U}_3 \end{pmatrix}$$

where $\boldsymbol{U}_i$ is a $n_i \times n_i$ matrix such that

$$\boldsymbol{U}_i \cdot \mathbf{1}_{n_i} = (\sqrt{n_i}, \underbrace{0, \ldots, 0}_{n_i - 1 \text{ times}})^t$$

with $\mathbf{1}_{n_i}$ a $n_i \times 1$ vector with all entries equal to 1. An appropriate candidate for $\boldsymbol{U}_i$ is the Householder transformation.

We have that $\boldsymbol{T}$ is congruent to $\boldsymbol{R}$:

$$\boldsymbol{U} \cdot \boldsymbol{R} \cdot \boldsymbol{U}^t = \boldsymbol{T} = \begin{pmatrix} \boldsymbol{T}_{11} & \boldsymbol{T}_{12} & \boldsymbol{T}_{13} \\ \boldsymbol{T}_{21} & \boldsymbol{T}_{22} & \boldsymbol{T}_{23} \\ \boldsymbol{T}_{31} & \boldsymbol{T}_{32} & \boldsymbol{T}_{33} \end{pmatrix}$$

where $\boldsymbol{T}_{pp}$ is a $n_p \times n_p$ matrix such that

$$\begin{aligned} \boldsymbol{T}_{pp}(i,i) &= 1 - \rho_{pp} + \rho_{pp} n_p\,,\, 1 \le i \le n_p \\ \boldsymbol{T}_{pp}(i,j) &= 0\,,\, 1 \le i \ne j \le n_p \end{aligned}$$

and $\boldsymbol{T}_{pq}$ is a $n_p \times n_q$ matrix such that

$$\begin{aligned} \boldsymbol{T}_{pq}(1,1) &= \rho_{pq}\sqrt{n_p n_q}, \\ \boldsymbol{T}_{pq}(i,j) &= 0 \text{ else }. \end{aligned}$$

Using an adequate permutation matrix $\boldsymbol{P}$, we have that $\boldsymbol{V}$ is congruent to $\boldsymbol{T}$:

$$\boldsymbol{P} \cdot \boldsymbol{T} \cdot \boldsymbol{P}^t = \boldsymbol{V} = \begin{pmatrix} \boldsymbol{X} & \mathbf{0}_{3\times n_1} & \mathbf{0}_{3\times n_2} & \mathbf{0}_{3\times n_3} \\ \mathbf{0}_{n_1\times 3} & \boldsymbol{W}_1 & \mathbf{0}_{n_1\times n_2} & \mathbf{0}_{n_1\times n_3} \\ \mathbf{0}_{n_2\times 3} & \mathbf{0}_{n_2\times n_1} & \boldsymbol{W}_2 & \mathbf{0}_{n_2\times n_3} \\ \mathbf{0}_{n_3\times 3} & \mathbf{0}_{n_3\times n_1} & \mathbf{0}_{n_3\times n_2} & \boldsymbol{W}_3 \end{pmatrix}$$

where $\boldsymbol{X}$ is a 3×3 matrix :

$$\boldsymbol{X} = \begin{pmatrix} (n_1-1)\rho_{11}+1 & \sqrt{n_1 n_2}\rho_{12} & \sqrt{n_1 n_3}\rho_{13} \\ \sqrt{n_1 n_2}\rho_{12} & (n_2-1)\rho_{22}+1 & \sqrt{n_2 n_3}\rho_{23} \\ \sqrt{n_1 n_3}\rho_{13} & \sqrt{n_2 n_3}\rho_{23} & (n_3-1)\rho_{33}+1 \end{pmatrix},$$

$\boldsymbol{W}_i$ is a $n_i \times n_i$ diagonal matrix with diagonal elements all equal to $1-\rho_{ii}$ and $\mathbf{0}_{u,v}$ is a $u \times v$ matrix with all entries equal to 0.

We then have nonnegative eigenvalues if :

- $\rho_{ii} \leq 1$
- the eigenvalues of the matrix $\boldsymbol{X}$ are nonnegative.

We therefore only have to make sure that the eigenvalues of $\boldsymbol{X}$ are nonnegative.

Let $\boldsymbol{F}$ be a 3×3 diagonal matrix with diagonal elements $(1/\sqrt{n_1}, 1/\sqrt{n_2}, 1/\sqrt{n_3})$. We have that $\boldsymbol{X}$ and $\boldsymbol{Z}$ are congruent :

$$\boldsymbol{F} \cdot \boldsymbol{X} \cdot \boldsymbol{F}^t = \boldsymbol{Z} = \begin{pmatrix} \rho_{11} + \frac{1-\rho_{11}}{n_1} & \rho_{12} & \rho_{13} \\ \rho_{12} & \rho_{22} + \frac{1-\rho_{22}}{n_2} & \rho_{23} \\ \rho_{13} & \rho_{23} & \rho_{33} + \frac{1-\rho_{33}}{n_3} \end{pmatrix}.$$

So we have to compute the eigenvalues of $\boldsymbol{Z}$ and make sure that they are nonnegative. This is equivalent to verifying that all principal determinants of $\boldsymbol{Z}$ are nonnegative.

We would like to obtain sufficient conditions for positive semidefiniteness for all values of n_1, n_2 and n_3.

After some algebra (see appendix B for details), we obtain the following sufficient conditions:

$$\begin{aligned}
\rho_{11}\rho_{22}\rho_{33} + 2\rho_{12}\rho_{13}\rho_{23} &\geq \rho_{11}\rho_{23}^2 + \rho_{22}\rho_{13}^2 + \rho_{33}\rho_{12}^2 \\
\rho_{11}\rho_{22} &\geq \rho_{12}^2 \\
\rho_{11}\rho_{33} &\geq \rho_{13}^2 \\
\rho_{22}\rho_{33} &\geq \rho_{23}^2 \\
\rho_{11} &\geq 0 \\
\rho_{22} &\geq 0 \\
\rho_{33} &\geq 0.
\end{aligned}$$

Not imposing these conditions may lead to impossible situations. Assume indeed the following parameters :

$$\begin{aligned}
\mu_1 = \mu_2 = \mu_3 &= 1 \\
\sigma_1 = \sigma_2 = \sigma_3 &= 1 \\
\rho_{11} = \rho_{22} = \rho_{33} &= 0.1 \\
\rho_{12} = \rho_{13} &= 0.1 \\
\rho_{23} &= 0.2
\end{aligned}$$

Then the sufficient conditions are not fulfilled for all values of n_1, n_2 and n_3 because

$$\rho_{22}\rho_{33} = 0.01 < 0.04 = \rho_{23}^2.$$

For $n_1 = n_2 = n_3 = 5$, the matrix is positive semidefinite whereas for $n_1 = n_2 = n_3 = 10$ it is not. It is however clear that the model must be valid for any size of the portfolio.

This clearly shows that working with multivariate models accounting for dependencies is a very complicated task. It also shows that accounting for dependencies between lines of business may not be coherent as soon as these lines of business increase in size.

5 Numerical Application

The required solvency level is given by the Tail Value at Risk at level ϵ of the random variable

$$S = \underbrace{X_1 + \cdots + X_{n_1}}_{S_1} + \underbrace{X_{n_1+1} + \cdots + X_{n_1+n_2}}_{S_2} + \underbrace{X_{n_1+n_2+1} + \cdots + X_{n_1+n_2+n_3}}_{S_3}.$$

S is normally distributed with parameters

$$\begin{aligned}\mu_S &= \sum_{i=1}^{3} n_i \mu_i \\ \sigma_S^2 &= \sum_{i=1}^{3} n_i \sigma_i^2 + 2\sum_{i=2}^{3} n_i n_1 \rho_{1i} \sigma_1 \sigma_i + 2 n_2 n_3 \rho_{23} \sigma_2 \sigma_3 + \sum_{i=1}^{3} n_i (n_i - 1) \rho_{ii} \sigma_i^2.\end{aligned}$$

S_i is normally distributed with parameters

$$\begin{aligned}\mu_{S_i} &= n_i \mu_i \\ \sigma_{S_i}^2 &= n_i \sigma_i^2 + n_i (n_i - 1) \rho_{ii} \sigma_i^2.\end{aligned}$$

The covariance between S_i and S_j is given by

$$\mathbb{C}ov(S_i, S_j) = n_i n_j \rho_{ij} \sigma_i \sigma_j.$$

The correlation between S_i and S_j is given by

$$\rho_{S_i,S_j} = \frac{n_i n_j \rho_{ij}}{\sqrt{n_i(1 + (n_i - 1)\rho_{ii})}\sqrt{n_j(1 + (n_j - 1)\rho_{jj})}}.$$

The total premium charged by the insurance company is $P = P_1 + P_2 + P_3$ including a margin equal to

$$Margin = P - \mu = \sum_{i=1}^{3} \alpha_i n_i \mu_i.$$

Assume the following parameters :

$$\begin{aligned}\mu_1 = \mu_2 = \mu_3 &= 1 \\ \sigma_1 = \sigma_2 = \sigma_3 &= 1 \\ \rho_{11} &= 0.1 \\ \rho_{22} &= 0.1 \\ \rho_{33} &= 0.1 \\ \rho_{12} &= -0.01 \\ \rho_{13} &= -0.01 \\ \rho_{23} &= 0.01 \\ \alpha_1 = \alpha_2 = \alpha_3 &= 0.10 \\ \epsilon &= 99\% \\ \text{Cost of capital} &= 15\% \\ \text{Available capital} &= 100.\end{aligned}$$

One easily checks that the sufficient conditions for positive semidefiniteness are verified.

We now look for the optimal number of risks to write in order to maximize the value creation for the shareholders. We find :

LOB	n	RAC	Margin	EVA	RORAC
1	94	37.09	9.4	3.84	25.34%
2	80	31.75	8	3.24	25.20%
3	79	30.95	7.9	3.26	25.53%
Total	253	99.79	25.3	10.33	25.35%

Table 5.1: Initial portfolio.

LOB 1 is favoured because it is negatively correlated with LOB 2 and LOB 3.

Now we change the loading of LOB 1 from 10% to -1%. We also change $\rho_{12} = \rho_{13}$ from -0.01 to -0.02 (sufficient conditions are still verified). We obtain

LOB	n	RAC	Margin	EVA	RORAC
1	24	-0.37	-0.24	-0.19	65.72%
2	93	50.16	9.3	1.78	18.54%
3	93	50.16	9.3	1.78	18.54%
Total	210	99.96	18.36	3.37	18.37%

Table 5.2: LOB 1 with loading -1% and more negatively correlated with LOB 2 and LOB 3.

We observe that it remains interesting to write risks from LOB 1 even though their margin is negative. Indeed these risks allow for an excellent diversification within the conglomerate and they in fact lend capital to the other LOB's. We also conclude that the negative EVA for LOB 1 is not informative. The RORAC for this LOB is extremely difficult to analyze.

Let us compute the EVA for the case where we would abandon LOB 1 :

LOB	n	RAC	Margin	EVA	RORAC
1	0	0	0	0	
2	90	49.76	9	1.54	18.09%
3	90	49.76	9	1.54	18.09%
Total	180	99.51	18	3.07	18.09%

Table 5.3: LOB 1 disregarded.

Clearly the value creation is now lower ($3.07 < 3.37$) than with LOB 1 within the conglomerate.

Now let us come back to the initial situation but assume the available capital is 200 instead of 100 :

LOB	n	RAC	Margin	EVA	RORAC
1	194	73.54	19.4	8.37	26.38%
2	165	63.06	16.5	7.04	26.17%
3	165	63.06	16.5	7.04	26.17%
Total	524	199.66	52.4	22.45	26.24%

Table 5.4: Increased available capital.

Table 5.4 shows that the value creation for the shareholder is now more than twice the value creation with the available capital equal to 100. This is obviously due to the diversification benefit. Because the risks are not perfectly dependent (comonotonic) there is a relative gain in terms of consumed capital.

Now let us come back to the initial situation and let us assume that the risks of LOB 1 have the following adapted characteristics : $\mu_1 = 2$ and $\sigma_1 = 2$:

LOB	n	RAC	Margin	EVA	RORAC
1	45	38.35	9	3.25	23.47%
2	79	31.19	7.9	3.22	25.33%
3	78	30.39	7.8	3.24	25.67%
Total	202	$99,93$	24.7	9.71	24.72%

Table 5.5: LOB 1 with larger risks.

Table 5.5 illustrates that the value creation is now lower than with the initial portfolio because the large risks consume a lot of capital. There is less diversification.

Finally, let us come back to the initial situation and assume that the conglomerate buys unlimited stop-loss reinsurance in excess of d. It is well known that reinsurance is an alternative to holding economic capital. It is therefore interesting to analyze the decision of buying reinsurance.

The pure premium for such reinsurance is given by

$$PP^{Re} = \sigma_S \phi\left(\frac{d-\mu_S}{\sigma_S}\right) - (d-\mu_S)\left(1-\Phi\left(\frac{d-\mu_S}{\sigma_S}\right)\right).$$

Assume a reinsurance loading β. The commercial reinsurance premium is then $P^{Re} = (1+\beta)PP^{Re}$. Let us compute the stop-loss premium for a threshold $d = VaR_{99\%}(S)$. We denote the retention of the insurance company by S^{Ret}. We now compare the EVA and RORAC for different loadings charged by the reinsurer. We do not analyze the EVA per line of business because we would have to resort to simulations to do so and it is part of our conclusion that looking at EVA per LOB is not optimal. In order to keep things simple, we exclude the possible default of the reinsurer. Note that the risk of default of the reinsurer is positively dependent with the risk of hitting the reinsurance cover. This would immediately lead to complicated models. We find

$$\begin{aligned}
n_1 &= 94 \\
n_2 &= 80 \\
n_3 &= 79 \\
P &= 278.3 \\
Margin(S) &= 25.3 \\
d = VaR_{99\%}(S) &= 362.17 \\
PP^{Re} &= 0.1590 \\
P^{Re} &= (1+\beta)PP^{Re} \\
VaR_{99\%}(S^{Ret}) &= 362.17 \\
TVaR_{99\%}(S^{Ret}) &= 362.17 \\
RAC(S^{Ret}) &= 362.17 - (278.3 - P^{Re}) \\
Margin(S^{Ret}) &= 25.3 - \beta PP^{Re} \\
EVA(S^{Ret}) &= Margin(S^{Ret}) - kRAC(S^{Ret}) \\
RORAC(S^{Ret}) &= \frac{Margin(S^{Ret})}{RAC(S^{Ret})}.
\end{aligned}$$

β	No reinsurance	1	4	9	14	19
$RAC(S^{Ret})$	99.79	84.2	84.68	85.47	86.27	87.06
$EVA(S^{Ret})$	10.33	12.51	11.96	11.04	10.13	9.21
$RORAC(S^{Ret})$	25.35%	29.86%	29.13%	27.92%	26.74%	25.59%

Table 5.6: Performance with reinsurance.

We observe that even with a substantial loading charged by the reinsurer, it may be interesting to buy the reinsurance cover. Now we may do even better. Indeed, we have some capital left thanks to the reinsurance. For that reason, it is also meaningless to draw conclusions on basis of EVA. RORAC is a good tool in this case. What we now are going to do is to optimize the number of policies to write knowing that we buy the reinsurance cover. We obtain the table 5.7 :

β	No reinsurance	1	4	9	14	19
n_1	94	113	112	111	110	109
n_2	80	95	95	94	93	92
n_3	79	95	95	94	93	92
$d = VaR_{99\%}(S)$	362.18	432.64	431.23	427.00	422.77	418.55
$RAC(S^{Ret})$	99.79	99.72	99.97	99.97	99.95	99.91
$EVA(S^{Ret})$	10.33	15.15	14.45	13.22	12.02	10.83
$RORAC(S^{Ret})$	25.35%	30.20%	29.45%	28.23%	27.03%	25.85%

Table 5.7: Optimal underwriting with reinsurance.

We conclude that the underwriting decision is influenced by the reinsurance cover bought by the insurer.

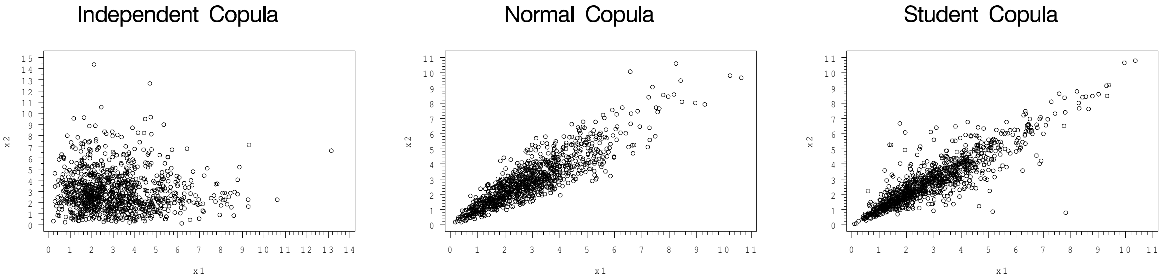
6 Extensions to the Multivariate Normal Model

In practice aggregated insurance risks may not be normally distributed. Often they present fatter tails than the tails of normal distributions. A possible extension of the multivariate normal distribution is then to work with multivariate elliptic distributions. See e.g. Valdez and Chernih (2003) for recent discussions in an actuarial context.

Another option is to resort to copulas. A copula is a function that links the distribution functions of different random variables within a stochastic dependency context. Copulas have been introduced a long time ago by Sklar (1959) and are gaining interest in actuarial science to model stochastic dependencies. The interested reader may consult e.g. Nelsen (1999) for a reference on copulas and Frees and Valdez (1998) for an introduction in an actuarial context. See also Embrechts et al. (2003). Summarizing we can say that modelling marginal distributions together with copulas provides a model for the aggregate portfolio accounting for dependencies between lines of business.

Copulas are excellent means in order to make sensitivity analyses about the stochastic dependency and their implications on e.g. capital allocation matters. However, they remain computationnally difficult to use in multiple dimensions. Therefore one may not be able to provide a full modelling like the multivariate normal model presented in the previous section. The best one can do with copulas is to modelize the stochastic dependency between lines of business.

Copulas can also cope with tail dependencies. The following graphs provide scatter plots of two types of claims with different dependency structures but identical gamma distributed marginals. The first dependency structure is independence, the second is based on the gaussian copula and the third on the Student copula.



Independent Copula

Normal Copula

Student Copula

Figure 6.1: Copulas.

We observe that the normal copula is not able to simulate simultaneous extremes whereas it is the case for the Student copula.

It is also worth noting that our socio-economic environment plays a major role in the determination of stochastic dependencies. When inflation is high, claim costs in workers compensation and motor third party liability are simultaneously high. On the other hand financial revenues increase, which is a partial hedge against the underwriting loss resulting from the high claim costs. When the economy is up, we have less fire, less default and less disability claims. We have the opposite when the economy is down. Therefore it is probably most interesting to develop economic environment generators allowing for a lot of stochastic dependencies within financial conglomerates.

7 Conclusion

We have analyzed that lines of business presenting negative margins may indeed be profitable for the whole conglomerate due to possible negative dependencies (or in fact low dependencies) between some lines of business. This goes against the traditional belief that managers must be remunerated according to the performance of their business unit. In fact, it may happen that the EVA of a particular LOB is negative. Nevertheless the manager should be rewarded if he has been able to reach the goals defined a priori in case these goals include writing policies of that LOB. Managers should be rewarded in function of the a priori objective, in our case the number of policies to write. This is true under the hypothesis that they write business at average market conditions. Then they should be rewarded in function of the global value creation. Only when managers beat the market they should be rewarded on a local view. This happens when a manager is able to obtain higher margins than the market. Once again, looking at a LOB individually without accounting for the possible dependencies with the financial conglomerate is not optimal. The decision to invest or to desinvest should be taken after a global calculation.

In section 3 we decided to disregard the effect of administrative expenses. However, it is clear that the decision to invest in one line of business or another cannot be disconnected from the effective administrative expenses incurred when writing these lines of business. The actual value creation for the shareholders is equal to Margin - capital charges - administrative expenses. The latter may have a dramatic influence on the decision to invest and requires a very close management. Indeed it would be stupid to apply very sophisticated models for the calculation of the margin and allocated capital when administrative expenses are roughly estimated.

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Appendix A : Results from Matrix Theory

Definition 1. A $n \times n$ symmetric matrix $\boldsymbol{M}$ is positive semidefinite if

$$\boldsymbol{x}^t \cdot \boldsymbol{M} \cdot \boldsymbol{x} \geq 0$$

for all nonzero $n \times 1$ vectors $\boldsymbol{x}$.

Theorem 2. *A $n \times n$ symmetric matrix $\boldsymbol{M}$ is positive semidefinite if and only if all its eigenvalues are nonnegative.*

Definition 3. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two $n \times n$ matrices. If there exists a nonsingular matrix $\boldsymbol{S}$ such that

$$\boldsymbol{B} = \boldsymbol{S} \cdot \boldsymbol{A} \cdot \boldsymbol{S}^t,$$

then $\boldsymbol{B}$ is said to be congruent to $\boldsymbol{A}$.

Definition 4. Let $\boldsymbol{A}$ be a symmetric $n \times n$ matrix. The inertia of $\boldsymbol{A}$ is the ordered triple

$$i(\boldsymbol{A}) = (i_+(\boldsymbol{A}), i_-(\boldsymbol{A}), i_0(\boldsymbol{A}))$$

where $i_+(\boldsymbol{A})$ is the number of positive eigenvalues of $\boldsymbol{A}$, $i_-(\boldsymbol{A})$ is the number of negative eigenvalues of $\boldsymbol{A}$ and $i_0(\boldsymbol{A})$ is the number of zero eigenvalues of $\boldsymbol{A}$.

Theorem 5. *Sylvester's law of inertia : Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two symmetric $n \times n$ matrices. There is a non-singular $n \times n$ matrix $\boldsymbol{S}$ such that $\boldsymbol{A} = \boldsymbol{S} \cdot \boldsymbol{B} \cdot \boldsymbol{S}^t$ if and only if $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same inertia, that is, the same number of positive, negative and zero eigenvalues.*

Definition 6. A $n \times n$ matrix $\boldsymbol{P}$ is called a permutation matrix if exactly one entry in each row and column is equal to 1, and all other entries are 0.
Multiplication by such matrices effects a permutation of the rows or columns of the object multiplied. Left multiplication of a matrix $\boldsymbol{A}$ by a permutation matrix $\boldsymbol{P}$ permutes the rows of $\boldsymbol{A}$ while right multiplication of a matrix $\boldsymbol{A}$ by a permutation matrix $\boldsymbol{P}$ permutes the columns of $\boldsymbol{A}$.

Theorem 7. *The determinant of a permutation matrix is either 1 or -1 so that permutation matrices are nonsingular.*

Theorem 8. *If $\boldsymbol{P}$ is a $n \times n$ permutation matrix, then*

$$\boldsymbol{P}^{-1} = \boldsymbol{P}^t.$$

Theorem 9. *Let $\boldsymbol{P}$ be a $n \times n$ permutation matrix. Since $\boldsymbol{P}^{-1} = \boldsymbol{P}^t$ permutes columns in the same way that the permutation matrix $\boldsymbol{P}$ permutes rows, the transformation $\boldsymbol{A} \rightarrow \boldsymbol{P} \cdot \boldsymbol{A} \cdot \boldsymbol{P}^t$ permutes the rows and columns of $\boldsymbol{A}$ in the same way.*

Definition 10. The Householder transformation is a $n \times n$ matrix $\boldsymbol{H}$ given by

$$\boldsymbol{H} = \boldsymbol{I}_n - 2\frac{\boldsymbol{v} \cdot \boldsymbol{v}^t}{\boldsymbol{v}^t \cdot \boldsymbol{v}}$$

where $\boldsymbol{I}_n$ is the $n \times n$ identity matrix and $\boldsymbol{e_1} = (1, \underbrace{0, \ldots, 0}_{n-1 \text{ times}})^t$.

The Householder transformation is such that

$$\boldsymbol{H} \cdot \boldsymbol{x} = \|x\|_2 \boldsymbol{e_1}$$

where $\|x\|_2 = \sqrt{\boldsymbol{x}^t \cdot \boldsymbol{x}}$.
Indeed we have

$$\begin{aligned} \boldsymbol{H} &= \boldsymbol{I}_n - 2\frac{\boldsymbol{v} \cdot \boldsymbol{v}^t}{\boldsymbol{v}^t \cdot \boldsymbol{v}} \\ \boldsymbol{H} \cdot \boldsymbol{x} &= \boldsymbol{x} - 2\frac{\boldsymbol{v}^t \cdot \boldsymbol{x}}{\boldsymbol{v}^t \cdot \boldsymbol{v}}\boldsymbol{v} \end{aligned}$$

If $\boldsymbol{v} = \boldsymbol{x} + \alpha \boldsymbol{e_1}$ then

$$\begin{aligned} \boldsymbol{v}^t \cdot \boldsymbol{x} &= \boldsymbol{x}^t \cdot \boldsymbol{x} + \alpha x_1 \\ \boldsymbol{v}^t \cdot \boldsymbol{v} &= \boldsymbol{x}^t \cdot \boldsymbol{x} + 2\alpha x_1 + \alpha^2 \\ \boldsymbol{H} \cdot \boldsymbol{x} &= \left(1 - 2\frac{\boldsymbol{x}^t \cdot \boldsymbol{x} + \alpha x_1}{\boldsymbol{x}^t \cdot \boldsymbol{x} + 2\alpha x_1 + \alpha^2}\right) \cdot \boldsymbol{x} - 2\alpha \frac{\boldsymbol{v}^t \cdot \boldsymbol{x}}{\boldsymbol{v}^t \cdot \boldsymbol{v}} \boldsymbol{e_1} \end{aligned}$$

where x_1 is the first entry of the vector $\boldsymbol{x}$.
Assume that $\alpha = -\|x\|_2$ then $\boldsymbol{H} \cdot \boldsymbol{x} = \|x\|_2 \boldsymbol{e_1}$.
In our case $\boldsymbol{x} = \mathbf{1}_n$ which implies $\|x\|_2 = \sqrt{n}$.
A Householder transformation is orthogonal :

$$\boldsymbol{H}^t \cdot \boldsymbol{H} = \boldsymbol{I}_n = \boldsymbol{H} \cdot \boldsymbol{H}^t,$$

implying that the Householder transformation is nonsingular.

Theorem 11. *Let $\boldsymbol{A}_i$ denote the $i \times i$ principal submatrix of $\boldsymbol{A}$ determined by the first i rows and columns of $\boldsymbol{A}$. A $n \times n$ symmetric matrix $\boldsymbol{A}$ is positive semidefinite matrix if and only if $det(\boldsymbol{A}_i) \geq 0$ for $i = 1, 2, \ldots, n$.*

Appendix B : Detailed Calculations for the Analysis of Positive Semidefiniteness of $\boldsymbol{Z}$

$$
\begin{aligned}
det(\boldsymbol{Z}_3) = {} & (1-\rho_{11})(1-\rho_{22})(1-\rho_{33})\frac{1}{n_1 n_2 n_3} \\
& +\rho_{33}(1-\rho_{11})(1-\rho_{22})\frac{1}{n_1 n_2} \\
& +\rho_{22}(1-\rho_{11})(1-\rho_{33})\frac{1}{n_1 n_3} \\
& +\rho_{11}(1-\rho_{22})(1-\rho_{33})\frac{1}{n_2 n_3} \\
& +(1-\rho_{11})(\rho_{22}\rho_{33}-\rho_{23}^2)\frac{1}{n_1} \\
& +(1-\rho_{22})(\rho_{11}\rho_{33}-\rho_{13}^2)\frac{1}{n_2} \\
& +(1-\rho_{33})(\rho_{11}\rho_{22}-\rho_{12}^2)\frac{1}{n_3} \\
& +2\rho_{12}\rho_{13}\rho_{23}+\rho_{11}\rho_{22}\rho_{33}-\rho_{11}\rho_{23}^2-\rho_{22}\rho_{13}^2-\rho_{33}\rho_{12}^2.
\end{aligned}
$$

$$
\begin{aligned}
det(\boldsymbol{Z}_2) = {} & (1-\rho_{11})(1-\rho_{22})\frac{1}{n_1 n_2} \\
& +\rho_{11}(1-\rho_{22})\frac{1}{n_1} \\
& +\rho_{22}(1-\rho_{11})\frac{1}{n_2} \\
& \rho_{11}\rho_{22}-\rho_{12}^2.
\end{aligned}
$$

$$
\begin{aligned}
det(\boldsymbol{Z}_1) = {} & (1-\rho_{11})\frac{1}{n_1} \\
& +\rho_{11}.
\end{aligned}
$$