

# Integrated Risk Measurement for Portfolio of Various Assets at Continuous Time Horizons <sup>\*</sup>

**Ng Kah Hwa** <sup>†</sup>

Risk Management Institute  
National University of Singapore, Singapore

**Ma Lanfang** <sup>‡</sup>

Department of Mathematics, Faculty of Science  
National University of Singapore, Singapore

March 17, 2007

---

<sup>\*</sup>The authors are grateful to Mr. Zong Jianping for discussing some technological details.

<sup>†</sup>Ng Kah Hwa, Director, Risk Management Institute, National University of Singapore, Block S16, Level 5, 6 Science Drive 2, Singapore 117546. Tel: (65) 6516-1011, Fax: (65) 6874-5430, e-mail: rm-ingkh@nus.edu.sg

<sup>‡</sup>Ma Lanfang, a PhD student, e-mail: g0301231@nus.edu.sg

# Integrated Risk Measurement for Portfolio of Various Assets at Continuous Time Horizons

## Abstract

Different financial products usually have very different risk profiles. In the financial industry, risk measures based on VaR for financial products are either dominant market VaR or credit VaR or Add VaR, which is obtained by evaluating market VaR and credit VaR separately and then add them together. The regulatory capital required by regulators is then computed according to the VaR, which will either underestimate or overestimate the products risks.

In order to reasonably measure market risk and credit risk together, in this study we present a new framework, with which we can measure integrated market risk and credit risk for portfolios consisting of various assets through continuous time horizons. Using Monte Carlo simulation, we employ this framework to portfolios consisting of bonds, stocks and bonds plus stocks with normal distributed asset return assumptions. We find that term structures of market VaR, credit VaR, integrated VaR and Add VaR are different for bond portfolio, stock portfolio and mixed portfolio, with the largest integrated VaR values for stock portfolio, the smallest ones for bond portfolio and those for mixed portfolio between them. Besides the type of assets, initial rating of the objective portfolio is also an important factor to determine the integrated VaRs. In this study, we also compare the integrated VaRs for portfolios with Student  $t$  and Skew  $t$  distributed asset returns to those with Normal distributed asset returns. We find that the integrated VaR magnitudes followed the pattern with  $Skew\ t > Student\ t > Normal$  for VaR at confidence level of 99% and 99.9%, and a contrary pattern for VaR95. This is caused by the different shapes of these distributions, among them Skew  $t$  distributions have the fattest left tails while Normal distribution has thinnest left tail, and the tail attributes are inherited by the portfolio value distribution.

This simulation study shows that asset type, initial rating, time horizon and asset return distribution assumptions are all significant factors to influence the portfolio value distributions and hence the integrated VaRs.

## KEY WORDS:

Integrated risk, Continuous time horizon, Portfolio of various assets, Skew  $t$  distribution, Term structure of Value-at-Risk

# 1 Introduction

In the last three decades, studies on credit risk modeling have made great progress. At first credit risk models mainly aimed to deal with individual defaults. There are two classes of such models: structural models originated by Black and Scholes (1973) and Merton (1974), and reduced form models (or intensity based models), which were originally developed by Jarrow and Turnbull (1995) and extended by Duffie and Singleton (1999) among others.

In recent years, the finance industry has realized that large losses are often caused by default contagion, so measuring portfolio of credit risks has become a popular research topic, and the popularity has been boosted by the issue of new Basel II regulatory requirements. Under Basel II, financial institutions are encouraged to build sound internal rating based (IRB) models for measuring their market risk, credit risk and operational risks, and then the regulatory capital can be calculated accordingly.

There are several industry sponsored Credit Value-at-risk models, two representatives of them are CreditMetrics proposed by JP Morgan (1997) and CreditRisk+ initiated by Credit Suisse Financial Products (CSFP)(1997). CreditMetrics which is based on the structural model whereby the default correlations are captured through one factor Gaussian copula. In contrast, CreditRisk+ is based on reduced form model and focused on default only, in which the default intensity is assumed to follow an exogenous Poisson process, and the default correlations are captured by the correlated Poisson diffusions.

These models are well constructed with clever insights about credit risks, and have been adopted by many smaller institutions who cannot afford to establish their own models. Their prevalence definitely makes for the progress of further research and application of more advanced credit risk models. However, these models share some common drawbacks. For example, they assume deterministic risk free rate, credit spreads and risk exposures. This assumption won't damage too much for bonds and loans in a relative stable market, but it is meaningless to swaps and other interest rate derivatives, because with this assumption their values would always be zero. In addition, because these models only consider the risks due to credit events, they segmented the closely related credit risk and market risk, and show incomplete risk profiles for the products. The correlation between market risk and credit risk is often difficult to determine. In the industry, one approach assumes perfect correlation, and adds the separately estimated market VaR and credit VaR together, which will lead to a conservative estimate. Another approach focuses on the product's dominant risk only, for example, credit risk for bonds and market risk for stocks. But for some products, such as Interest Rate Swap and Credit Default Swap, their credit risk and market risk can affect to each other by trading with different credit

quality counterparts, which will lead to economic capital arbitrage if only dominant risk is considered. Hence, the segmentation of credit risk and market risk will lead to some danger in accurate economic capital allocations.

In the last five years, many researchers have developed integrated risk models which can evaluate closely related market risk and credit risk synthetically. Some researchers tried to remedy the drawbacks for the above discussed models. Among them, Kiesel, Peraudin and Taylor(2003) introduced stochastic rating specific credit spreads into CreditMetrics framework while keeping risk-free rate deterministic. They found that the spread fluctuations were the major contribution to the VaR values of high credit quality portfolios. Extending their work, Grundke(2005) introduced both stochastic risk-free rate and credit spreads into CreditMetrics . Based on the standardized asset return, the author took the rating transition risk, credit spread risk, interest rate risk and recovery risk all correlated, and then evaluated these risks synthetically for a large homogeneous bond portfolio. He showed that if the stochastic nature of the risk factors are neglected ,underestimation of risks happened, and it was particularly serious for high credit quality portfolios with low asset return correlations.

Some other researchers proposed new integrated models. For example, Barnhill and Maxwell (2002) developed a model which not only included stochastic interest rate and credit spreads, but also simulated a set of 24 equity market indices representing various economy sectors. With all these simulated factors constituting the future financial environments, this model could produce reasonable transition probability matrix, and with this newly generated transition probability matrix other than historical one, this model could measure the portfolio VaR accurately. Based on intensity based model, Kijima and Muromachi (2000)proposed a model which had correlated stochastic interest rate and default intensity processes, which could not only produce no arbitrage bond prices but capture the different term structures for default intensities over different credit ratings. But it can not capture the rating migration information. Jobst and Zenios (2001) incorporated elements both from rating based models and from stochastic intensity models in their framework, and then extended applications to portfolios consisting of interest rate and credit risk sensitive products.

Most of these models mainly apply to fixed income portfolios except Medova and Smith (2005) and Tanaka and Muromachi (2003). Medova and Smith (2005) measured integrated risks for a foreign exchange forward contract at different time horizons. Although the authors illustrated their model with a very simple example - an individual foreign exchange contract, they did show us the meaning of measuring integrated risks . Tanaka and Muromachi (2003) extended the model of Kijima and Muromachi (2000). They not only included stochastic interest rate and default intensity processes, but also introduced stochastic diffusions of stock prices and foreign exchange rates in it, but it also

inherited the drawbacks of the original model which has been discussed before.

In general, all these integrated risk models are based either on structural models or on intensity models. Compared with industry sponsored credit risk models, most of them have taken the stochastic nature of interest rate and credit spreads into account. However, they have some insufficiencies to be a realistic integrated risk model. First, most of them focused on integrated risk for fixed income portfolios, while few models studied on portfolios consisting of various assets, such as bonds, loans, stocks, swaps and foreign exchange products etc. This is due to the fact that different assets have totally different risk profiles, and it is difficult to measure their correlations and their different kinds of risks in a uniform framework. Second, these models only investigated the integrated risks at time horizon of 1 year, and no other time horizons are considered. This is unreasonable since integrated models usually measure the portfolios' market risk and credit risk simultaneously, but the time horizon for market risk (usually 1 day or 10 working days) and for credit risk (usually 1 year) are very different. So realistic integrated models should consider the effects of different time horizons on portfolio VaRs. Third, for structural based models, they assumed that the asset returns are normally distributed, except that Grundke (2005) also investigated the portfolio risks with Student t distributed asset returns. Indeed, normal distribution has some good properties such as having analytical solutions, but asset return has been verified to be non-normal based on research in the past three decades. This assumption will affect the accuracy of portfolio risk evaluation. Fourth, as discussed above, approach one (measuring market risk and credit risk separately and then add them together) overestimates portfolio risk and approach two (dominant risk only) underestimates portfolio risk, but the underestimation and overestimation are not totally investigated, especially how the underestimation and overestimation change with the increase of time horizon.

So the main aim of this study is to develop a new framework which could fulfill the following functions. First, it could measure the market risk and credit risk for portfolios consisting of various assets synthetically, and could take their default correlations into consideration adequately. Second, it could measure integrated risks for portfolios at any time horizon, and could check the changes of underestimation or overestimation with respect to different time horizons. Third, it should agree with the fact that asset returns are non-normal distributed.

Up to now, it is the most complete framework for integrated risk measurement. It can measure integrated risks for portfolios consisting of various assets, and this capability should be important to banks, insurers and pension funds since their investments are often diversified with various kinds of financial products. In addition, it takes into account the non-normal attribute of asset returns and the effects of different time horizons, which should generate more reasonable risk values and show clearer risk profiles, and will lead

to a better economic capital allocation strategy.

This paper is organized as follows. Section 2 discusses the methodology in which the new framework will be introduced in detail. Section 3 gives the numerical results and corresponding discussions . Section 4 provides the conclusion.

## 2 Methodology

In this section, the model proposed by Grundke(2005)was set as a benchmark since it has some good attributes. For example, it has considered the stochastic nature of interest rate, credit spreads and recovery rate, and it also included the rating migration information. However, it considers only the integrated risks for a large homogeneous portfolio consisting of only zero coupon bonds at time time horizon of 1 year. In this study, we make three extensions from this benchmark model to establish a new framework. The first extension is to consider the model for continuous time horizons. The second extension is to evaluate integrated risks for portfolios consisting of various assets. The last extension is to introduce the asymmetric and fat-tailed asset return distributions into the new framework. The benchmark model and its three extensions are described below.

### 2.1 Review of benchmark model

In the benchmark model, with the standardized asset return as the base, the credit risks ( including downgrade risk, default risk and recovery risk) and market risk (including interest rate risk and credit spread risk) for a large homogenous portfolio are synthetically evaluated. This portfolio consists of N exchangeable zero coupon bonds issued by N different companies with identical initial ratings. The bonds' face value(F), maturity (T) and the pairwise correlation ( $\rho_v$ ) among the firms asset returns are all identical.

#### 2.1.1 Modeling downgrade risk and default risk

The downgrade risk and default risk come from the uncertainty that at which rating bond n will stay at the future time horizon H. In this benchmark model, the bond n's future rating can be determined by the firm's standardized asset return  $X_n$ .

$$X_n = \sqrt{\rho_v - \rho_{rv}^2} Z + \rho_{rv} X_r + \sqrt{1 - \rho_v} \varepsilon_n \quad (\rho_{rv}^2 \leq \rho_v, n \in \{1, \dots, N\}) \quad (1)$$

Where common factor Z, interest rate factor  $X_r$  , firm specific factors  $\varepsilon_1, \dots, \varepsilon_N$  are independent N(0,1) distributed random variables.  $\rho_{rv}$  is the identical correlation between all

asset returns and interest rate. Hence, all the  $N$  firms asset returns  $\mathbf{X}$  follow multivariate normal distribution (or Gaussian distribution), to distinguish it from other asset return distributions which will be referred later, hereafter it is denoted as  $\mathbf{X}^G$ . Then

$$\begin{cases} E(\mathbf{X}^G) &= \mathbf{0} \\ COV(\mathbf{X}^G) &= \Sigma \end{cases} \quad (2)$$

where

$$\Sigma = \begin{pmatrix} 1 & \rho_v & \dots & \rho_v \\ \rho_v & 1 & \dots & \rho_v \\ \dots & \dots & \dots & \dots \\ \rho_v & \rho_v & \dots & 1 \end{pmatrix}$$

With the normal distributed asset returns assumption, future rating for bond  $n$  can be

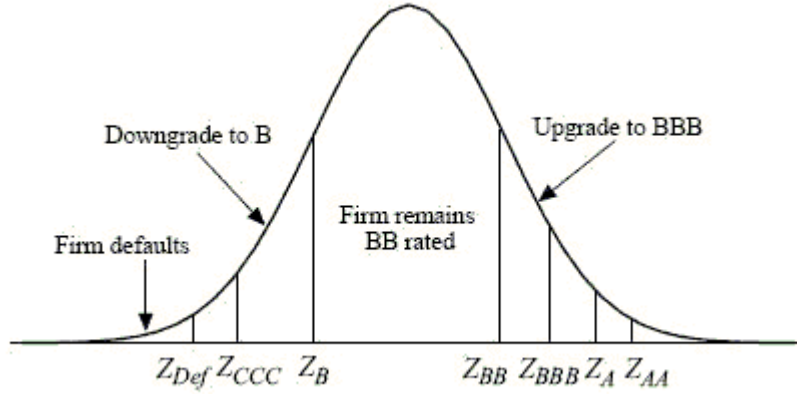


Figure 1: *Asset return Distribution and rating thresholds, adapted from CreditMetrics-Technical Document*

determined according to the methodology of CreditMetrics. Now consider a bond with initial rating BB, at the end of one year, its rating can stay in any of the  $k$  states ( $k \in \{1, \dots, 8\}$ ) with 1 representing the best rating AAA and 8 representing the worst rating. As illustrated in Figure 1, to determine the future rating of bond  $n$ , we need to compare the  $N(0, 1)$  distributed random variable  $X_n^G$  and the rating thresholds ( $Z_{Def}, Z_{CCC}, \dots, Z_{AA}$ ) to see which interval the  $X_n^G$  falls into. The thresholds  $\mathbf{Z}_{(1 \times 7)}$  is derived from a one-year transition matrix  $Q = (q_{ik})_{8 \times 8}$  which is published by rating agencies, such as Moody's or Standard and Poor's. Since the initial rating of this bond is BB, which corresponds to  $i = 5$ , only the fifth row of  $Q = (q_{ik})_{8 \times 8}$  is used in this example. The detailed derivation procedure is illustrated in Table 1. The thresholds for other ratings can be determined in a similar way. With the threshold matrix ( $Z$ ) $_{7 \times 7}$  and the simulated asset returns  $X_n^G$ , the future ratings and then the downgrade risk and default risk for bonds can be determined.

Table 1: Determining the rating thresholds

$X_n$	Future rating	Prob. $q_{5k}$	threshold $Z_{5k}$
$X_n \leq Z_{Def}$	default	$\Phi(Z_{Def})$	$Z_{Def} = \Phi^{-1}(q_{5Def})$
$Z_{Def} \leq X_n \leq Z_{CCC}$	CCC	$\Phi(Z_{CCC}) - \Phi(Z_{Def})$	$Z_{CCC} = \Phi^{-1}(q_{5CCC} + q_{5Def})$
$Z_{CCC} \leq X_n \leq Z_B$	B	$\Phi(Z_B) - \Phi(Z_{CCC})$	$Z_B = \Phi^{-1}(q_{5B} + q_{5CCC} + q_{5Def})$
$Z_B \leq X_n \leq Z_{BB}$	BB	$\Phi(Z_{BB}) - \Phi(Z_B)$	$Z_{BB} = \Phi^{-1}(\sum_{k=BB}^{k=Def} q_{5k})$
$Z_{BB} \leq X_n \leq Z_{BBB}$	BBB	$\Phi(Z_{BBB}) - \Phi(Z_{BB})$	$Z_{BBB} = \Phi^{-1}(\sum_{k=BBB}^{k=Def} q_{5k})$
$Z_{BBB} \leq X_n \leq Z_A$	A	$\Phi(Z_A) - \Phi(Z_{BBB})$	$Z_A = \Phi^{-1}(\sum_{k=A}^{k=Def} q_{5k})$
$Z_A \leq X_n \leq Z_{AA}$	AA	$\Phi(Z_{AA}) - \Phi(Z_A)$	$Z_{AA} = \Phi^{-1}(\sum_{k=AA}^{k=Def} q_{5k})$
$X_n \geq Z_{AA}$	AAA	$1 - \Phi(Z_{AA})$	- - -

### 2.1.2 Modeling interest rate risk

In the benchmark model, the stochastic risk-free rate evolves as an Ornstein-Uhlenbeck process with constant coefficients as proposed by Vasicek (1977).

$$dr(t) = k[\theta - r(t)]dt + \sigma dW(t), \quad r(0) = r_0 \quad (3)$$

where  $r_0$ ,  $k$ ,  $\theta$  and  $\sigma$  are positive constants.  $r(t)$  is normally distributed with mean  $\theta + (r(0) - \theta)e^{-kt}$  and volatility  $\sqrt{\frac{\sigma_r^2}{2k}(1 - e^{-2kt})}$ , and the closed form solution for this stochastic differential equation is

$$r(t) = \theta + (r(0) - \theta)e^{-kt} + \sqrt{\frac{\sigma_r^2}{2k}(1 - e^{-2kt})} X_r \quad (4)$$

where  $X_r$  is the same notation as in Equation (1). The forward rate  $FR(X_r, H, T)$  is needed when calculating the bond value at time horizon  $H$ . It can be derived by

$$FR(X_r, H, T) = -\frac{1}{T - H} \left( \left( \frac{1}{k}(1 - e^{-k(T-H)}) (R(\infty) - (\theta + (r(0) - \theta)e^{-kH} + \sqrt{\frac{\sigma_r^2}{2k}(1 - e^{-2kH})} X_r) ) - (T - H)R(\infty) - \frac{\sigma_r^2}{4k^3}(1 - e^{-k(T-H)})^2 \right) \right) \quad (5)$$

where  $R(\infty) = \theta + \lambda \frac{\sigma_r}{k} - \frac{\sigma_r^2}{2k^2}$  denotes the return of default-free zero coupon bonds with infinite maturity, and  $\lambda$  is the market price of interest rate risk.

### 2.1.3 Modeling credit spread risk

The rating specific credit spreads  $S_k(H, T)$  ( $k \in \{1, \dots, 7\}$ ) are assumed to follow multivariate  $N(\mu_k, \sigma_k^2, R)$  distribution, where  $\mu_k$  and  $\sigma_k$  are the means and volatilities of the annual credit spread rates, and the correlation matrix among credit spreads is  $R$ . In the benchmark model, the credit spreads are assumed to be determined jointly by common factor  $Z$ , interest rate factor  $X_r$  and the specific credit rating factor  $\eta_k$ , and the time horizon  $H$  is fixed as 1 year.

$$S_k(H, T) = \mu_k + \sigma_k( \rho_{rs}X_r + \rho_{zs}Z + \sqrt{1 - \rho_{rs}^2 - \rho_{zs}^2} \eta_k ) \quad (k \in 1, \dots, 7) \quad (6)$$

where  $\eta_k$ s are correlated standard normally distributed random variables with correlation matrix  $\tilde{R}$ , which will be needed to simulate  $S_k(H, T)$ . The correlated  $\eta$  can be generated by Cholesky decomposition of  $chol(\tilde{R}) \times \xi$  where  $\xi$  are i.i.d  $N(0,1)$  distributed random variables.

$$\begin{aligned} COV\left(\frac{S_i - \mu_i}{\sigma_i}, \frac{S_j - \mu_j}{\sigma_j}\right) &= R_{ij} = \rho_{sz}^2 + \rho_{sr}^2 + (1 - \rho_{sz}^2 - \rho_{sr}^2)COV(\eta_i, \eta_j) \Rightarrow \\ COV(\eta_i, \eta_j) &= \frac{R_{ij} - (\rho_{sz}^2 + \rho_{sr}^2)}{1 - \rho_{sz}^2 - \rho_{sr}^2} \Rightarrow \\ \tilde{R}_{ij} &= \frac{R_{ij} - (\rho_{sz}^2 + \rho_{sr}^2)}{1 - \rho_{sz}^2 - \rho_{sr}^2} \end{aligned} \quad (7)$$

In fact, from Kiesel, Perraudin and Taylor (2003) we can see that the mean, volatilities, and correlations of  $S_k(H, T)$  also changed with different maturities or T-H, but the effects of T-H are small enough to be ignored when compared with other risk sources.

### 2.1.4 Modeling recovery risk

In the benchmark model, in the case bond n defaults at time horizon  $H$ , its recovery rate  $\delta_n$  is assumed to be a beta-distributed random number, which will be drawn individually to ensure the independence across different exposures. The first two moments of recovery rates are set to match the historical statistical data. This is also the practical model that is used in the industry. An alternative method referred to in Grundke (2005) is to model the recovery rate as a log-normally distributed random number

$$\delta_n = e^{\mu_n + \sigma_n R_n} \quad (8)$$

with

$$R_n = \alpha_n Z + \beta_n X_r + \gamma_n X_n + \sqrt{1 - \alpha_n^2 - \beta_n^2 - \gamma_n^2} \eta_n$$

where  $\alpha_n, \gamma_n$  and  $\sigma_n \in \mathbb{R}_+$ ,  $\mu_n$  and  $\beta_n \in \mathbb{R}$  and  $\alpha_n^2 + \beta_n^2 + \gamma_n^2 \leq 1$ .

With all the risk factors properly modeled, then the bond value at time horizon H can be calculated

$$\begin{cases} v^k(H, T) = F e^{-(FR(X_r, H, T) + S_k(H, T))(T-H)} & \text{bond survives} \\ v^8(H, T) = \delta_n p(H, T) & \text{bond defaults} \end{cases}$$

The portfolio value at time horizon H is the sum of N bond values.

$$\Pi = \sum_{n=1}^N v_n^k(H, T) \quad (k \in \{1, \dots, 8\}) \quad (9)$$

From the above statements, we can see that in the benchmark model, all the down-grade risk, default risk, credit spread risk and recovery risk are all affected by common factor Z and interest rate factor  $X_r$ , which implies that these risks are correlated. Thus in this way, the market risk and credit risk for the bond portfolio can be integrated.

## 2.2 Extension to continuous time horizon (H)

In the benchmark model, the time horizon H is fixed as 1 year. However a continuous-time modeling framework is very useful because it can evaluate the integrated risks at arbitrary points in time. We follow the continuous-time Markov Chain model in Schönbucher(2003) to extend the benchmark model into a new continuous time horizon framework.

As commonly used in reduced form models, the hazard rate is modeled as a Poisson process. Similarly the rating transition intensities can also be modeled in this way . The transiting probability from rating k to rating l in a small time interval  $\Delta t$  is assumed to be proportional to  $\Delta t$ :

$$P[R(t + \Delta t) = l | R(t) = k] = \lambda_{kl} \Delta t \text{ for } k \neq l$$

The probability of staying in rating k is:

$$P[R(t + \Delta t) = k | R(t) = k] = I - \sum_{l \neq k} \lambda_{kl} \Delta t = I + \lambda_{kk} \Delta t$$

where  $\lambda_{kk} = - \sum_{l \neq k} \lambda_{kl}$  .

For small time intervals, the transition probability matrix  $Q(t, t + \Delta t)$  can be approximated by a Taylor series:

$$Q(t + \Delta t) = I + \Delta t \Lambda(t) + \text{terms of order } (\Delta t)^2 \quad (10)$$

where  $I$  is identity matrix and  $\Lambda = (\lambda_{kl})$  with  $(k, l) \in \{1, \dots, 8\}$  is the matrix of transition intensities, or also known as the generator matrix.

For a large time interval  $[t, s]$ , it is subdivided into  $i$  subintervals of length  $\Delta t$ . Since the rating transition processes are assumed to have attributes of Markov property and time homogeneity, the transition probability Matrix for large time interval  $[t, s]$  can be calculated as

$$Q(t, s) = Q(t, t + i\Delta t) = (I + \Delta t\Lambda)^i = \left(I + \frac{s-t}{i}\Lambda\right)^i$$

In the limiting case, it becomes:

$$Q(t, s) = e^{\{(s-t)\Lambda\}} \quad (11)$$

The exponential algorithm for matrix can be fulfilled by

$$e^x = I + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots \quad (12)$$

From equation (11) we can see that if the generator matrix  $\Lambda$  is known, the transition matrix  $Q(t, s)$  can be derived by replacing  $x$  with  $(s-t)\Lambda$  in Equation (12). Schönbucher(2003) also reviewed several approaches to derive generator matrix  $\Lambda$ . In this study, we directly take advantage of the generator matrix  $\Lambda$  published by Standard and Poor's directly.

Table 2: Approximate generator matrix published by S&P, adapted from Schönbucher(2003)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	-11.59	10.75	0.42	0.13	0.29	0.00	0.00	0.00
AA	0.95	-10.61	8.32	0.81	0.26	0.27	0.00	0.00
A	0.08	3.24	-12.14	7.46	0.90	0.40	0.00	0.06
BBB	0.06	0.36	7.56	-17.75	7.91	1.40	0.13	0.33
BB	0.04	0.22	0.58	8.85	-26.12	12.95	1.36	2.08
B	0.00	0.21	0.27	0.47	6.40	-19.98	5.90	6.73
CCC	0.00	0.04	1.44	1.36	2.46	10.13	-43.53	28.10
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

With this generator matrix  $\Lambda$ , the calculated one year transition probabilities agree with the historical average one-year rating transition frequencies published by S&P, which is shown in Table 3.

Table 3: Historical average one-year rating transition probabilities (1981-1991)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	89.10	9.63	0.78	0.19	0.30	0.00	0.00	0.00
AA	0.86	90.10	7.47	0.99	0.29	0.29	0.00	0.00
A	0.09	2.91	88.94	6.49	1.01	0.45	0.00	0.09
BBB	0.06	0.43	6.56	84.27	6.44	1.60	0.18	0.45
BB	0.04	0.22	0.79	7.19	77.64	10.43	1.27	2.41
B	0.00	0.19	0.31	0.66	5.17	82.46	4.35	6.85
CCC	0.00	0.00	1.16	1.16	2.03	7.54	64.93	23.19
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100

With this S&P published generator matrix, we can get the transition probability matrix  $Q(t, s)$  at any time point  $t$ , and then the rating transition thresholds. Hence the credit risks of the bond portfolio can be determined at continuous time horizons. The market risk evaluations for this bond portfolio also can be extended to continuous time horizons. Since the market risk factors are normally distributed with their means proportional to  $H$  and volatilities proportional to the square root of  $H$ , the evaluation of risks due to market changes of interest rate and credit spreads can be easily extended to continuous time horizons.

## 2.3 Extension to portfolio consisting of various assets

The integrated risks for portfolios consisting of various assets are not easy to be evaluated, because different assets usually have very different risk profiles and pricing formulas, and the correlations among various assets are difficult to capture. To tackle these problems, we take the standardized asset returns as a base. This is because most of the assets suffer market risk and credit risk simultaneously, usually the credit risks are determined by counterpart's asset return when based on structural models, and the market risks for some kinds of assets are also affected by their firms' asset returns. In the following parts, we try to introduce various assets into the portfolio one by one.

### 2.3.1 Stocks

Traditionally, the stock price follows a geometric Brownian motion (GBM), and the dynamics of the stock price process  $S$  is

$$dS = \mu_s S dt + \sigma_s S dW_t$$

Then the stock price has closed form solution

$$S_t = S_0 e^{(\mu_s - \frac{1}{2}\sigma_s^2)t + \sigma_s \epsilon \sqrt{t}} \quad (13)$$

where  $\mu_s$  and  $\sigma_s$  are the expected return and volatility for stock price respectively.

Because stocks are actively traded products, usually only the dominant risk - market risk is considered through time horizon of one day or ten working days. When a firm defaults, its stock price will jump to near zero, so stocks also suffer credit risk. However, the default risks for stocks are often ignored in the industry, which will lead to underestimation of stock risks, especially for long term investments.

Now we consider a portfolio consisting of N assets coming from N different companies, the assets are either exchangeable bonds or exchangeable stocks. The pairwise correlations among different asset returns  $\rho_v$  are also assumed to be identical. In this portfolio, the bond prices can be determined following the method discussed above, and the stock prices can be determined by

$$S_t = \begin{cases} S_0 e^{(\mu_s - \frac{1}{2}\sigma_s^2)t + \sigma_s X_n^G \sqrt{t}} & \text{no default} \\ 0 & \text{default} \end{cases} \quad (14)$$

Where  $X_n^G$  is the same as that in Equation(1).

Comparing Equation (13) and Equation (14), we found that the default risks for stocks are introduced, and the more important point is that the N(0,1) distributed random variable  $\epsilon$  in Equation (13) is replaced by standardized normally distributed asset return  $X_n^G$ . This replacement is very important, because through which the default correlations and correlations of price movements between any two assets (two bonds, a bond and a stock, and two stocks) are captured adequately. This replacement also guarantees that the stock return dynamics always keep in the same direction with asset return dynamics, which agree with the industry practice that the unobserved asset returns are usually approximately by observed equity returns.

### 2.3.2 Swaps

For some products, such as bonds and loans, only one counterparty suffers potential default risk. But for some other products, such as interest rate swaps(IRS), forward rate agreements(FRA), credit default swaps(CDS) etc., both of the two counterparties suffer from potential default losses. We show how to include these assets into the portfolio illustrated with an interest rate swap.

An interest rate swap is worth zero when it is initiated between two counterparties A and B. Subsequently its value may become positive or negative. Counterparty A suffers from default loss only when the interest rate swap has positive value to it. If the swap has negative value, then counterparty B has potential loss caused by defaults of A.

We follow Bomfim (2002) to price interest rate swaps. The value of its fixed leg is

$$V_{fix}(t) = \sum S\delta_i P(t, T_i)$$

and the value of its floating leg

$$V_{fl}(t) = 1 - P(t, T)$$

then the swap value at time t

$$V_{swap}(t) = V_{fix}(t) - V_{fl}(t) = \sum S\delta_i P(t, T_i) - 1 + P(t, T) \quad (15)$$

where S is the swap rate, which is determined at initial time to make the swap value zero. Assuming counterparty A receives fixed rate and pays floating LIBOR rate, the swap value at time t to counterparty A is

$$V_{swap}(t) \begin{cases} = 0 & \text{if } V_{swap}(t) > 0 \text{ and } B \text{ defaults} \\ = V_{fix}(t) - V_{fl}(t) & \text{if } V_{swap}(t) > 0 \text{ and } B \text{ survives} \\ = V_{fix}(t) - V_{fl}(t) & \text{if } V_{swap}(t) < 0 \end{cases} \quad (16)$$

When the swap value is positive to counterparty A, it suffers potential loss of this positive value caused by default of counterparty B, while if the swap value is negative to A, then B suffers the potential loss caused by default of A. The contract value of swap will jump to zero if either counterparty defaults, the default probabilities can be derived from the counterparties' asset returns based on structure models. The market risks of swaps are derived from volatilities of LIBOR rates.

### 2.3.3 Foreign exchange products

If financial institutions invest both in domestic market and international market, the risks of exchange rate should be considered. We define the spot exchange rate  $F_x$  as a geometric Brownian motion process, in a risk neutral world the process is

$$dF_x = (r - r_f)F_x dt + \sigma_x F_x dW_t \quad (17)$$

where  $r$  is the domestic risk-free rate,  $r_f$  is the foreign risk-free rate, and  $\sigma_x$  is the exchange rate's volatility. The log-normally distributed spot exchange rate  $F_x(t)$  has closed form solution. With determined exchange rate  $F_x$ , at any time  $t$ , we can get any foreign exchange product's value based on domestic currency, for example, value of foreign exchange forward, currency swap and stocks bought in international markets.

## 2.4 Extension to non-normal distributed asset returns

In the benchmark model, asset returns are modeled as multivariate normal distributed random variables, but asset return distribution has been verified to be non-normal based on the research in past three decades. In the new framework, we try to capture the asymmetry and fat-tail attributes of asset return distribution. We simulate Student  $t$  and skew  $t$  distributed asset returns and then compare them with normal distributed asset returns.

### 2.4.1 Student $t$ distributed asset returns

Student  $t$  distribution has fatter tails than normal distribution. The multivariate Student  $t$  distributed random numbers can be generated by:

$$\mathbf{X}^T = \mu + \sqrt{W}AZ \quad (18)$$

where

- (1)  $\mathbf{Z} \sim N_k(\mathbf{0}, I_k)$
- (2)  $W$  is a positive r.v. with inverse gamma distribution, ie,  $\mathbf{W} \sim Ig(\frac{v}{2}, \frac{v}{2})$ , or equivalently,  $v/W \sim \chi_v^2$
- (3)  $A \in \mathbb{R}_{d \times k}$  and  $\mu \in \mathbb{R}^d$  are constant matrix and vector, and  $AA' = \Sigma$

The joint density function of multivariate  $\mathbf{X}^T$  is given by:

$$f(\mathbf{x}^T) = \frac{\Gamma(\frac{v+d}{2})}{\Gamma(\frac{v}{2})(\pi v)^{\frac{d}{2}}} \left| \Sigma \right|^{\frac{1}{2}} \left( 1 + \frac{(\mathbf{x}^T - \mu)' \Sigma^{-1} (\mathbf{x}^T - \mu)}{v} \right)^{-\frac{(v+d)}{2}} \quad (19)$$

The mean vector and covariance matrix  $\mathbf{X}^T$  are:

$$\begin{cases} E(\mathbf{X}^T) &= \mu \\ COV(\mathbf{X}^T) &= \frac{v}{v-2} \Sigma \text{ existed for } v > 2 \end{cases} \quad (20)$$

Recall that the multi-normal asset returns  $\mathbf{X}^G$  have mean  $\mathbf{0}$  and covariance matrix  $\Sigma$ . To be comparable, we need to match the first two moments of the asset return

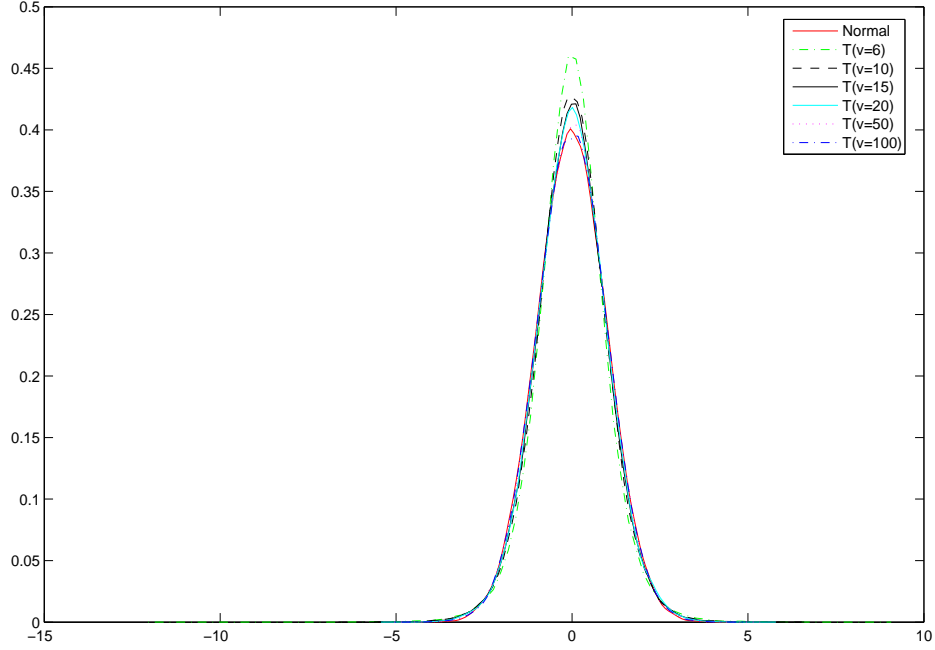


Figure 2: Comparison with Normal and Student  $t$  distributions

distributions. This can be guaranteed by

$$\begin{aligned}
 \mathbf{X}_n^T &= \sqrt{\frac{v-2}{v}} \sqrt{W} \mathbf{X}_n^G \\
 &= \sqrt{\frac{v-2}{v}} \sqrt{W} (\sqrt{\rho_v - \rho_{rv}^2} Z + \rho_{rv} X_r + \sqrt{1 - \rho_v} \varepsilon_n)
 \end{aligned} \tag{21}$$

We can verify that  $E(\mathbf{X}^T) = \mathbf{0}$  and  $COV(\mathbf{X}^T) = \Sigma$ .

In this study we take the degrees of freedom  $v$  of Student  $t$  distribution as 6, 10, 15, 20, 50 and 100, and then compare their distribution shapes, and especially the tails. From Figure 2 we can see that Student  $t$  distributions are symmetric and would converge to a standard normal distribution with big degrees of freedom  $v$ . In addition, all the student  $t$  distributions have larger kurtosis and fatter tails than those of normal distribution.

We focus on the enlarged left tails of these distributions and shown in Figure 3. Among the seven distributions, the Student  $t$  distribution with  $v = 6$  has the fattest left tail and

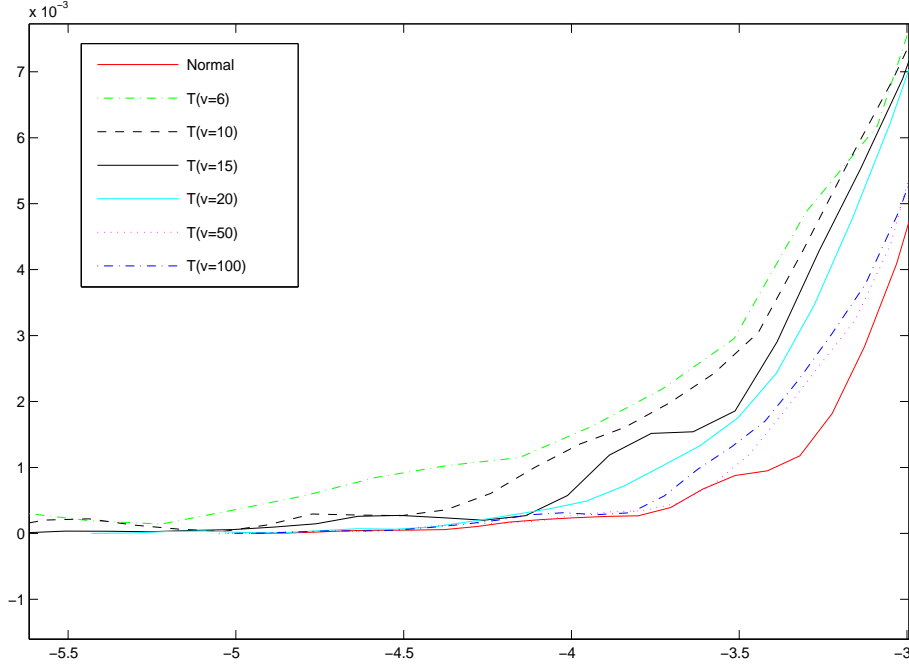


Figure 3: *Left tails of Normal and Student t distributions*

normal distribution has the thinnest tail. For the rest of the distributions, the left tails become fatter with the increase of  $v$ , so that when  $v$  goes to infinity, Student  $t$  distribution will be the same as normal distribution.

#### 2.4.2 Skew $t$ distributed asset returns

Although Student  $t$  distributions can capture the fat tails of asset returns, they are symmetric and cannot reflect the skewness of asset returns, so we turn to skew  $t$  distribution. Skew  $t$  distribution has been studied and utilized in finance by some researchers recently, such as McNeil, Frey and Embrechts (2005) and Hu (2005). The skew  $t$  distributed asset returns can be generated by

$$\mathbf{X}^{\text{st}} = \mu + W\gamma + \sqrt{W}AZ \quad (22)$$

where  $\mu$  and  $\gamma$  are parameter vectors in  $\mathbf{R}^d$ .  $\mu$ ,  $W$ ,  $A$  and  $Z$  are identical to those in the definition of Student  $t$  distribution in Equation (18), but the new parameter vector  $\gamma$  is introduced to reflect the skewness, and if  $\gamma = 0$ , the skew  $t$  distribution coincides with

Student t distribution.

The joint density function of multivariate Skew t distribution is given by

$$f(\mathbf{X}^{\text{st}}) = c \frac{K_{\frac{v+d}{2}} \sqrt{(v+Q_x)(\gamma \Sigma^{-1} \gamma)} e^{(\mathbf{x}^{\text{st}} - \mu)' \Sigma^{-1} \gamma}}{\left( \sqrt{(v+Q_x)(\gamma \Sigma^{-1} \gamma)} \right)^{\frac{-(v+d)}{2}} \left( 1 + \frac{Q_x}{v} \right)^{\frac{v+d}{2}}} \quad (23)$$

where  $Q_x = (\mathbf{x}^{\text{st}} - \mu)' \Sigma^{-1} (\mathbf{x}^{\text{st}} - \mu)$  and  $c = \frac{2^{1-\frac{v+d}{2}}}{\Gamma(\frac{v}{2})(\pi v)^{\frac{d}{2}} |\Sigma|^{\frac{1}{2}}}$

The mean and covariance of Skew t distributed random vector  $\mathbf{X}^{\text{st}}$  are:

$$\begin{cases} E(\mathbf{X}^{\text{st}}) &= \mu + \gamma \frac{v}{v-2} \\ COV(\mathbf{X}^{\text{st}}) &= \frac{v}{v-2} \Sigma + \gamma \gamma' \frac{2v^2}{(v-2)^2(v-4)}, \text{ exists when } v > 4 \end{cases} \quad (24)$$

The first two moments are also matched with those of Student t and normal distribution, that is,  $E(\mathbf{X}^{\text{st}}) = \mathbf{0}$  and  $COV(\mathbf{X}^{\text{st}}) = \Sigma$ . Equation (24) shows that  $COV(\mathbf{X}^{\text{st}}) > \Sigma$ , we need to transform  $X^{\text{st}}$  to:

$$\begin{aligned} X^{\text{st}} &= \sqrt{\alpha} (\mu + W\gamma + \sqrt{W} \tilde{A} \mathbf{Z}) \\ &= \sqrt{\alpha} \left( \mu + W\gamma + \sqrt{W} (\sqrt{\tilde{\rho}_v - \rho_{rv}^2} Z + \rho_{rv} X_r + \sqrt{1 - \tilde{\rho}_v} \varepsilon_n) \right) \end{aligned} \quad (25)$$

where  $\tilde{A}\tilde{A}' = \tilde{\Sigma}$ ,  $\tilde{\Sigma}$  is a N-by-N matrix with 1 as diagonal elements and  $\tilde{\rho}_v$  as off-diagonal elements. The unknown parameters  $\alpha$ ,  $\mu$  and  $\tilde{\rho}_v$  can be determined when parameters  $v$ ,  $\gamma$  and  $\Sigma$  are given.

$$\begin{cases} E(\mathbf{X}^{\text{st}}) = \mathbf{0} \Rightarrow \mu + \gamma \frac{v}{v-2} = 0 \\ COV(\mathbf{X}^{\text{st}}) = \alpha \left( \frac{v}{v-2} \Sigma + \gamma \gamma' \frac{2v^2}{(v-2)^2(v-4)} \right) = \Sigma \Rightarrow \\ \begin{cases} 1 = \alpha \left( \frac{v}{v-2} + \gamma \gamma' \frac{2v^2}{(v-2)^2(v-4)} \right) \\ \rho_v = \alpha \left( \frac{v}{v-2} r h o_v + \gamma \gamma' \frac{2v^2}{(v-2)^2(v-4)} \right) \end{cases} \end{cases} \quad (26)$$

then

$$\begin{cases} \mu &= -\gamma \frac{v}{v-2} \\ \alpha &= \frac{1}{\frac{v}{v-2} + \gamma \gamma' \frac{2v^2}{(v-2)^2(v-4)}} \\ \tilde{\rho}_v &= \rho_v - (1 - \rho_v) \gamma \gamma' \frac{2v}{(v-2)(v-4)} \end{cases} \quad (27)$$

With determined parameters  $\mu$ ,  $\alpha$ , and  $\tilde{\rho}_v$ , the skew t distributed asset returns  $\mathbf{X}^{\text{st}}$  can be simulated. As the portfolio is a large homogenous one, all the parameters for the N different assets are identical. To illustrate how the skewness parameters  $\gamma$  affect

the asset return distributions, we fixed  $v$  as 6 and varied  $\gamma$  from 0.1 to -0.3 step by -0.1 because Hu (2005) found that parameters  $\gamma$  for asset returns are usually negative near -0.2.

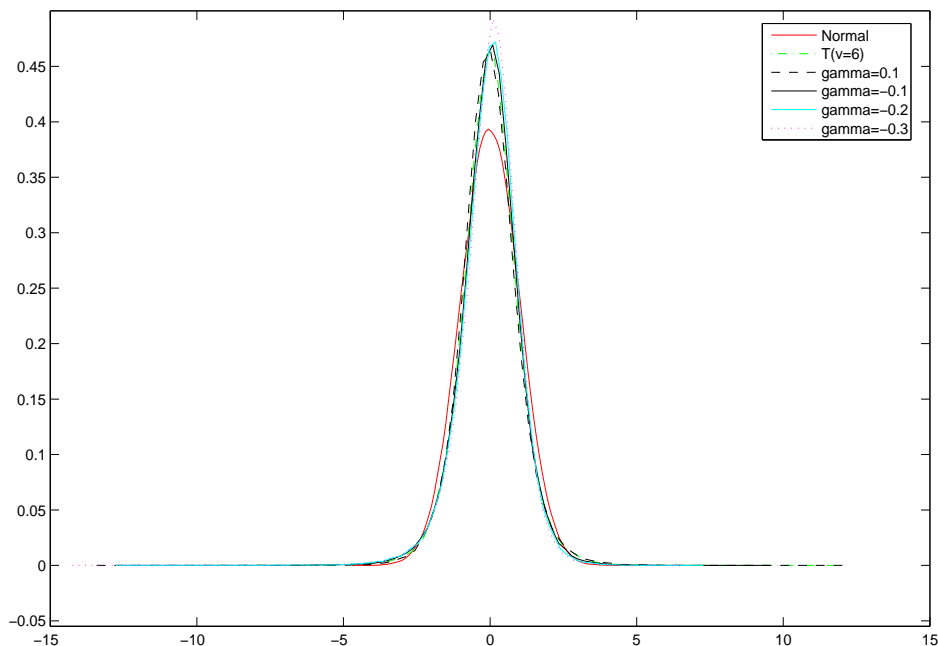


Figure 4: *Normal, Student t and Skew t distributions*

In Figure 4, Normal distribution, Student t distribution with  $v=6$ , and skew t distributions with  $v=6$  and  $\gamma = 0.1, -0.1, -0.2, -0.3$  respectively) are shown. All these distributions have identical first two moments-zeros and  $\sum$  respectively. In this figure we can see that all the Student t and skew t distributions have larger kurtosis and fatter tails than normal distribution. For left-skewed skew t distribution, which corresponds to negative  $\gamma$ , when  $\gamma$  changes from -0.1 to -0.3 step by -0.1, the distribution becomes more left-skewed and its left tail becomes fatter. In contrast to negative  $\gamma$ , the positive  $\gamma$  led to a right-skewed distribution with its right tail fatter than left tail.

Figure 5 shows the enlarged tails. We can see that for both sides the tails of Student t and skew t distributions are fatter than normal distribution. The left-skewed skew t distribution with  $\gamma = -0.3$  has the fattest left tail and thinnest right tail except for those of normal distribution, and vice versa for right-skewed distribution with  $\gamma = 0.1$ .

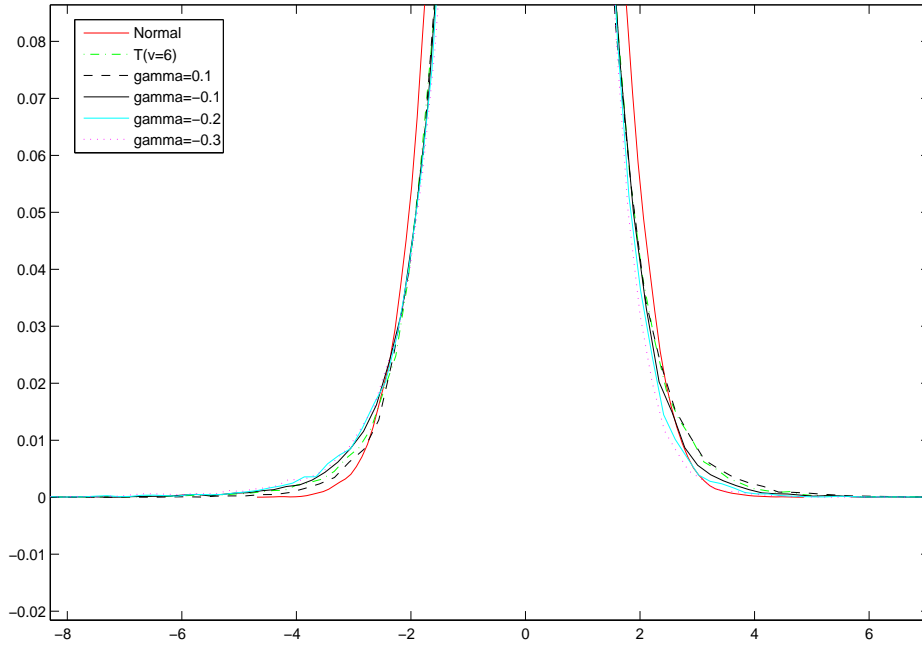


Figure 5: *Tails of Normal, Student  $t$  and Skew  $t$  distributions*

Under the non-normal asset return assumptions, the credit risk and market risk will be determined in a similar way as with the normal asset return assumptions.

### 3 Results and Discussions

Following the Methodology introduced in section 2, in this section a Monte Carlo simulation study is implemented with the path number  $M=500000$ . The first four moments and VaR values at confidence levels (denoted as CL hereafter) 95%, 99% and 99.9% for portfolio value distributions with different conditions and assumptions are calculated and then compared.

### 3.1 Benchmark case

The benchmark case is that the investor has 200 million USD and will invest all his money in a large homogeneous portfolio consisting of exchangeable zero coupon bonds issued by 200 different companies, and the time horizon is fixed as 1 year.

To be comparable, most of the parameters are consistent with those in Grundke(2005). All the parameters used in simulation are listed in Table 4.

Table 4: Specification of Parameters

Portfolio parameters			
Number of bonds(N)	N=200	FaceValue(F)	F=1
Maturity(T)	T=3	time Horizon(H)	H=1
asset correlation	$\rho_v = 0.2$	corr(asset, interest rate)	$\rho_{rv} = -0.05$
Risk free rate parameters			
k=0.4	$\theta=0.06$	$r_0=0.06$	$\sigma_r=0.01$
market price of risk	$\lambda=0.5$		
Credit spread parameters			
correlation( $X_r, S$ )	$\rho_{X_r, S} = -0.1$	correlation(Z,S)	$\rho_{Z, S} = -0.1$
Mean of $S_k$	$\mu_k = [35.6 \ 41.0 \ 58.2 \ 86.0 \ 189.6 \ 331.2 \ 1320]$		
Volatility of $S_k$	$\sigma_k = [14.3 \ 14.8 \ 21.5 \ 30.6 \ 74.0 \ 117 \ 480]$		
correlation Matrix	R(will be shown separately)		
Beta distributed recovery rate parameters			
Mean	$\mu_\delta = 0.538$	Volatility	$\sigma_\delta = 0.2686$

$$R = \begin{pmatrix} & \begin{array}{c|ccccccc} & AAA & AA & A & BBB & BB & B & CCC \end{array} \\ \begin{array}{c} AAA \\ AA \\ A \\ BBB \\ BB \\ B \\ CCC \end{array} & \begin{array}{ccccccc} 1 & 0.92 & 0.84 & 0.72 & 0.70 & 0.64 & 0.64 \\ 0.92 & 1 & 0.86 & 0.70 & 0.75 & 0.61 & 0.64 \\ 0.84 & 0.86 & 1 & 0.89 & 0.81 & 0.67 & 0.61 \\ 0.72 & 0.70 & 0.89 & 1 & 0.77 & 0.69 & 0.67 \\ 0.70 & 0.75 & 0.81 & 0.77 & 1 & 0.65 & 0.69 \\ 0.64 & 0.61 & 0.67 & 0.69 & 0.65 & 1 & 0.65 \\ 0.64 & 0.64 & 0.61 & 0.67 & 0.69 & 0.65 & 1 \end{array} \end{pmatrix}$$

With these parameters, the main statistical characteristics of portfolio value distributions are calculated and compared under three different situations. The first situation is that the portfolio suffers market risk only, which means the risks are from interest rate and credit spread changes, no rating transition happens during the year. The second situation is that the portfolio suffers credit risk only, which means the risks are from downgrade,

default and uncertainty of recovery rate, and with deterministic interest rate and credit spreads, i.e.,  $\sigma_r = 0$  and  $\sigma_k = [0 \ 0 \ 0 \ 0 \ 0 \ 0]$ . The last situation is that the portfolio suffers credit risk and market risk simultaneously, under which the integrated risks are measured.

Table 5: Portfolio VaRs for Market, Credit and Integrated risks

	Mean	STD	Skewness	kurtosis	VaR 95	VaR 99	VaR 99.9
Market risk only							
AA	214.91	2.4175	0.0279	2.9962	3.9557	5.5727	7.3758
BBB	215.8743	2.4737	0.0344	3.0026	4.0463	5.6881	7.5679
B	221.2628	4.1619	0.0519	3.0107	6.7821	9.524	12.5431
Credit risk only							
AA	213.0966	0.2443	-5.9063	65.7936	0.3730	1.0575	2.4174
BBB	213.3264	1.4350	-3.4802	23.8670	2.6839	5.8229	11.4097
B	211.821	7.7859	-1.6639	7.0531	15.4592	27.0715	42.9390
Integrated risk							
AA	214.762	2.4496	0.0179	3.026	4.0129	5.6288	7.5975
BBB	214.9797	3.0383	-0.5713	4.9404	5.0904	8.3287	14.3131
B	213.5481	9.1821	-1.1355	5.2807	17.3235	29.1833	44.7079

Table 5 reports the first four moments and VaR values at CL 95%,99% and 99.9% of portfolio value distributions. The results in the first three-row block are about the market risks for portfolios with initial rating AA, BBB and B respectively, the skewness are near zero and kurtosis are near three, which indicate that the portfolio value is almost normally distributed when only market risks are considered. In contrast, when only credit risks are considered with those results shown in second block, the skewness for AA, BBB and B rated portfolios are -5.9063, -3.4802 and -1.6639, which are all negative and significantly different from zero, and the corresponding kurtosis are 65.7936, 23.867 and 7.0531 respectively, which are significantly different from three. This implies that the portfolio value distributions are left-skewed and highly-peaked, especially for portfolio with initial rating AA. In the last block, the integrated risks are reported. Compared with market VaRs and credit VaRs, at all three confidence levels, the integrated VaRs are larger than either of them and smaller than the sum of them. However, for different initial rating portfolios, their integrated VaRs follow different patterns. For example, the integrated VaR at 99% confidence level(CL) for initially AA rated portfolio is 5.6288 , which is very close to its corresponding market VaR 5.5727 , while for initially B rated portfolio, the 99% CL integrated VaR is 29.1833, which is near to its credit VaR 27.0715. These trends are expected, since for high credit quality portfolio the market risk is dominant while for low credit quality portfolio the credit risk becomes important. And these findings

also agree with those in previous works done by Kiesel, Perraudin and Taylor(2003) and Grundke(2005).

As discuss in the Introduction section, there are two approaches to evaluate risks for various financial products in the industry. Approach A is to take dominant risk, and Approach B is to evaluate market VaR and credit VaR separately and then add them together . Approach A is said to underestimate risks and Approach B is said to overestimate risks. In this study, both the underestimation and overestimation effects are analyzed.

Table 6: Underestimation and Overestimation

	Market VaR VS Integrated VaR			Credit VaR VS Integrated VaR		
	VaR 95	VaR 99	VaR 99.9	VaR 95	VaR 99	VaR 99.9
AA	98.57%	99.00%	97.08%	9.29%	18.79%	31.82%
BBB	79.49%	68.30%	52.87%	52.72%	69.91%	79.72%
B	39.15%	32.64%	28.06%	89.24%	92.76%	96.04%

	Add VaR VS Integrated VaR		
	VaR 95	VaR 99	VaR 99.9
AA	107.87%	117.79%	128.90%
BBB	132.21%	138.21%	132.59%
B	128.39%	125.40%	124.10%

The results of underestimation are shown in the upper two blocks of Table 6. We can see that all the ratios are smaller than 1 although in different extents, which means for market risk and credit risk, whichever is taken as dominant risk and considered only, it does lead to underestimation of risks. This underestimation is especially serious for high credit quality portfolio if only credit risk is included, which can be verified by the ratios - credit VaR to integrated VaR, they are only 9.29%, 18.79% and 31.82% at CL 95, 99 and 99.9 respectively. If market risk is taken as dominant risk, then the most serious underestimation happens to low credit portfolios. In the third block of Table 6, the overestimation effects are listed. All figures are larger than 1 and the largest ratio occurs for the BBB rated portfolio, which means the Add-VaRs do overestimate portfolio risks, and the overestimation is most serious for middle rated bonds. One of the interpretations is that all integrated VaRs are larger than dominant VaRs and smaller than Add VaRs, for high credit quality portfolio, its credit VaR can be ignored when compared with its market VaR, so the ratio between Add VaR and integrated VaR cannot be too large. Similar conclusion can be derived for low credit quality portfolios due to their credit risk dominating their market risk. In contrast, for the middle rated portfolio at time horizon of 1 year, both its market risk and credit risk are large enough and neither can be ignored, its ratio

of Add VaR to integrated VaR might be larger than those with high or low credit qualities.

### 3.2 Time Horizon H

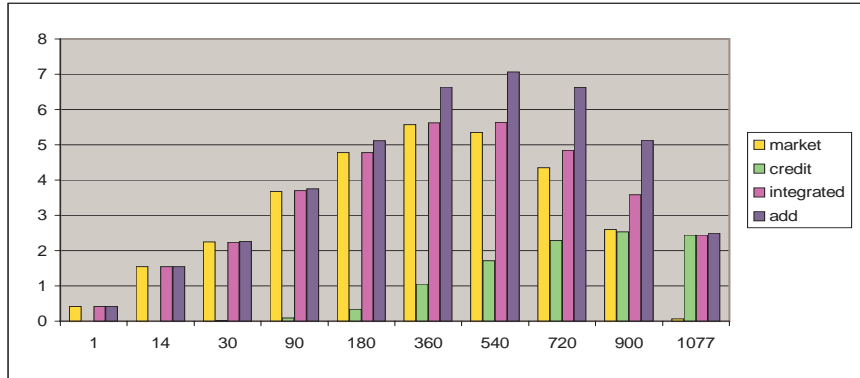
The benchmark case consists of portfolio VaRs at fixed time horizon of 1 year, while there are no reasons to fix H at 1 year for integrating market risk and credit risk because their time horizons are totally different. In this section, we consider continuous time horizon H and investigate how the portfolio VaRs changed with the increase of H, and whether the VaR term structures follows different patterns for portfolios with different initial ratings.

Ten time points are selected to approximate the continuous time horizon H, i.e., 1 day, 14 days, 1 month, 3 month, 6 month, 1 year, 1.5 year, 2 year, 2.5 year, and 3 years before 3 year - the maturity. All time points are expressed in days under the assumption that there are 360 days in a year and 30 days in every month. All market VaRs, credit VaRs, integrated VaRs and add VaRs are calculated at the ten time points, then the term structure of VaR values and VaR ratios at CL 99% are compared and analyzed.

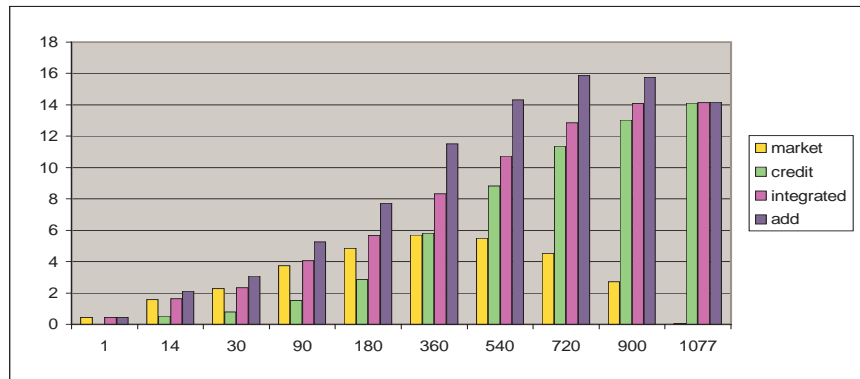
Figure 6 shows the term structure of market VaRs, credit VaRs, integrated VaRs and Add VaRs for portfolios with initial rating AA, BBB and B in panel (a), (b) and (c) respectively. In general, the term structures of market VaRs with different initial ratings are similar, the VaR values increase first and then decrease with the largest VaRs at time horizon of 360 days. The largest market VaRs for different initial ratings do not increase too much when the initial credit quality worsens from rating AA to rating B. One possible interpretation is that the bond value formula is  $v^k(H, T) = Fe^{-(FR(X_r, H, T) + S_k(H, T))(T-H)}$ , the market VaRs are determined by the volatility of  $FR(X_r, H, T) + S_k(H, T)$  and T-H, as we have verified that the volatilities of  $FR(X_r, H, T)$  and  $S_k(H, T)$  are both increasing function of H, but T-H is decreasing function of H, so the whole market VaR term structures hump in middle and decrease to near zero when H approaches to T. The market VaRs change a little for different rating portfolio, that is because they are influenced by identical  $FR(X_r, H, T)$  and slightly different  $S_k(H, T)$ .

In contrast, the term structures of credit VaRs followed different patterns. They are all increasing functions of time horizon H, but the increasing speeds are remarkably distinguishable for different credit quality portfolios. For AA, BBB and B rated portfolios, the largest credit VaRs were 2.4387, 14.1185 and 59.0256 respectively at time horizon of 1077 days. This agrees with the common knowledge that cumulative credit risks are always increase with time, and more defaults are expected to happen for poor credit quality portfolio than for good credit quality ones.

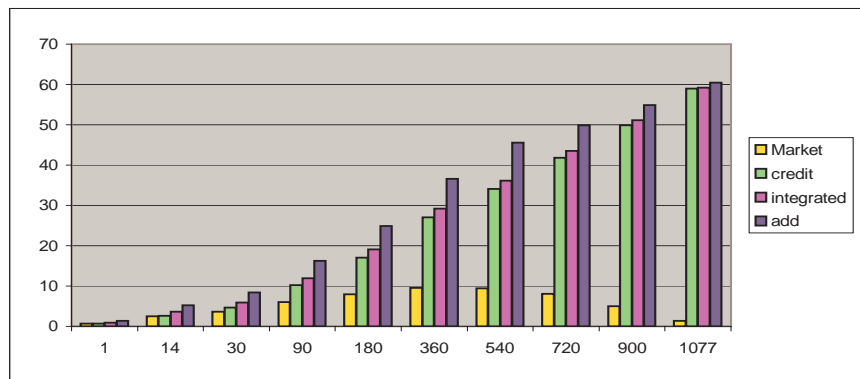
The term structures of integrated VaR and Add VaR are also shown in Figure 6, but



(a) VaRs for portfolio with initial rating AA

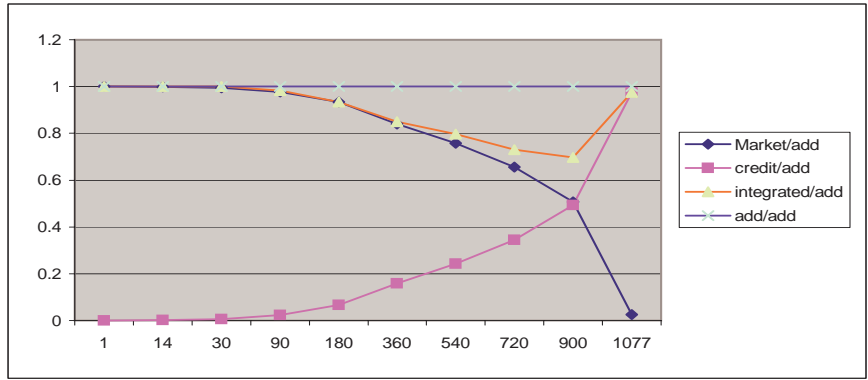


(b) VaRs for portfolio with initial rating BBB

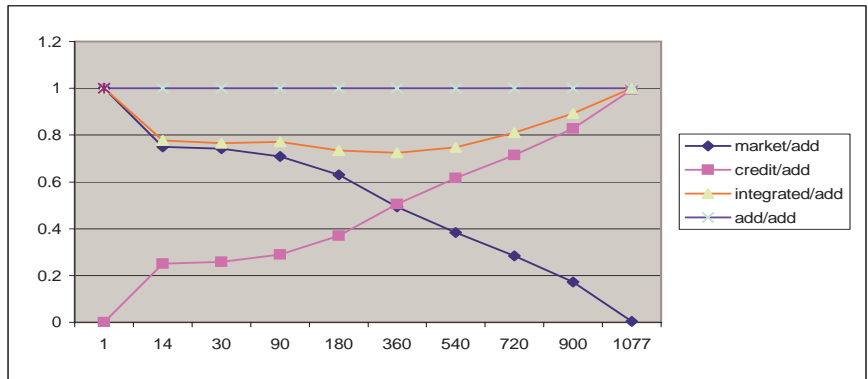


(c) VaRs for portfolio with initial rating B

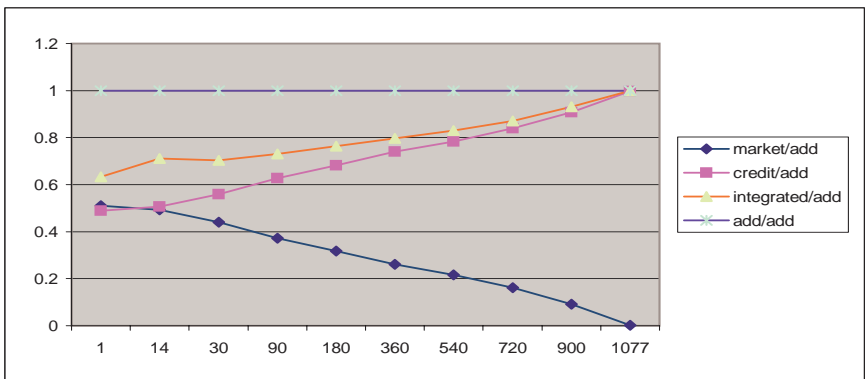
Figure 6: VaR Values for Portfolio with initial rating AA, BBB, and B



(a) VaRs ratios for portfolio with initial rating AA



(b) VaRs ratios for portfolio with initial rating BBB



(c) VaRs ratios for portfolio with initial rating B

Figure 7: VaR ratios for Portfolio with initial rating AA, BBB and B

they become clearer when combined with Figure 7. Figure 7 shows the term structures of percentages for market VaR VS add VaR, credit VaR VS add VaR, integrated VaR VS add VaR and add VaR VS add VaR for portfolios with different initial ratings. In panel (a), from H=1 days to 720 days, the term structure of integrated VaR follows that of market VaR closely, at H=900 days, the market VaR and credit VaR become half-half of the add VaR, after that the term structure of integrated VaR follows that of credit VaR. That is expected because for high credit quality portfolio, its market risk is dominant but with the increase of time, its credit risk becomes significant. Things are some what different to middle and low credit quality portfolios, in panel (b) the integrated VaR is jointly determined by market VaR and credit VaR, while in panel (c) the integrated VaR is determined mainly by credit VaR during the whole period. This is because for middle rated portfolio, its market risk and credit risk are both large enough and neither can be ignored, but for low credit quality portfolio, its credit risk is dominant especially when the time horizon becomes larger.

The overestimation of Add VaRs can be seen from the distance between term structures of integrated VaR and add VaR, the most serious overestimation for initial AA, BBB and B rated portfolios are at H=900, H=360 and H=1 day respectively. At those time points the market VaR and credit VaR are equally important, which implies that the perfect correlation assumption leads to most conservative estimates when the two kinds of risks are comparative.

### 3.3 Portfolio consisting of various assets

As discussed in the Methodology section, the new framework can deal with portfolios consisting of various assets. To maintain the computational tractability, without loss of generality, we focus on portfolios consisting of stocks and bonds only.

#### 3.3.1 Portfolio of stocks only

Before studying a mixed portfolio, we investigate the risk properties of stock portfolio first, especially its term structures of market VaR, integrated VaR and the underestimation when only the market risk is considered.

The stock portfolio consists of 200 exchangeable stocks issued by 200 different companies with equal weights. All the stocks have identical expected returns, volatilities, and their issue firms have identical initial rating. The expected return is fixed as  $\mu_s = 0.1$ , in fact we verified that the portfolio's market VaRs and integrated VaRs are all increasing function of  $\mu_s$ . Stock return volatility  $\sigma_s$  is varied from 0.1 to 0.5 stepped by 0.1, other

parameters are the same as described in section 3.1.

Traditionally, only market risks are considered for stocks. But actually they also suffer from credit risk, i.e., when the issuing firm defaults, the stock price will jump to zero. The integrated risk evaluation for stocks will consider the market risk and the default events simultaneously. Now we fix  $H=1$  year and examine how stock return volatility and default events influence the portfolio VaRs.

Table 7: Market and Integrated VaRs for stock portfolios with  $\mu_s = 0.1$  and  $H=1$

Market	mean	std	skew	kurtosis	VaR 95	VaR 99	VaR 99.9
0.1	221.05	10.00	0.134	3.012	16.085	22.238	28.777
0.2	221.05	20.03	0.271	3.121	31.346	42.641	54.402
0.3	221.04	30.10	0.416	3.303	45.816	61.132	76.757
0.4	221.03	40.25	0.547	3.540	59.532	78.218	96.479
0.5	221.07	50.65	0.696	3.860	72.495	93.630	113.970
Integrated	AA						
0.1	221.03	10.03	0.111	3.067	16.112	22.593	30.179
0.2	221.01	20.03	0.264	3.122	31.280	42.821	55.167
0.3	221.05	30.13	0.411	3.298	45.794	61.270	77.593
0.4	221.06	40.31	0.556	3.561	59.440	78.227	96.800
0.5	221.07	50.72	0.704	3.912	72.500	93.971	114.140
Integrated	BBB						
0.1	220.29	11.00	-0.234	3.661	18.331	28.222	42.876
0.2	220.49	20.73	0.144	3.232	32.943	47.061	64.228
0.3	220.61	30.58	0.346	3.311	46.913	64.543	84.201
0.4	220.86	40.63	0.523	3.550	60.368	80.397	102.080
0.5	220.88	50.77	0.685	3.890	73.032	95.254	117.000
Integrated	B						
0.1	208.61	21.44	-1.025	4.707	40.377	66.193	99.145
0.2	210.96	28.85	-0.425	3.502	50.805	77.517	109.700
0.3	212.86	36.95	-0.033	3.215	61.316	88.809	119.990
0.4	214.50	45.58	0.254	3.310	71.815	100.340	130.420
0.5	215.79	54.61	0.497	3.626	82.209	110.760	139.560

Table 7 shows the market risks and integrated risks for stock portfolio with initial rating AA, BBB and B in four blocks. We can see that the results in the second and third blocks are very close to those in the first block, that means for high credit quality stock portfolios, market risk is definitely dominant, and default events rarely happen in one

year period. In the third column of the upper three blocks are the standard deviations (denoted as std hereafter), they are proportional to the stock return volatilities. However, things are some what different for stock portfolio with initial rating B, its std are not proportional to  $\sigma_s$  any more. When considering integrated risks for this B initially rated portfolio, its std are 21.44, 28.85, 36.95, 45.58, and 54.61 respectively, which corresponds to  $\sigma_s = 0.1, 0.2, 0.3, 0.4$  and  $0.5$ . They are significantly larger than 10.00, 20.03, 30.10, 40.25 and 50.65, which correspond to the std when considering market risk only. The largest increase of std is 11.41 with  $\sigma_s = 0.1$  and the least increase of std is 3.89 with  $\sigma_s = 0.5$ . All the integrated VaRs are significantly larger than market VaRs, and their differences shrank with the increase of  $\sigma_s$ . That is because for stock portfolio with high initial credit qualities, the dominant market risks are determined by stock return volatilities. But for low credit quality stock portfolio, its default risk becomes relatively important compared with market risk, especially when the market risks become smaller due to the decrease of stock return volatilities.

Now we fixed the stock return volatility  $\sigma_s$  at 0.3 and investigate what the term structures look like for market VaR and integrated VaRs with initial rating AA, BBB and B.

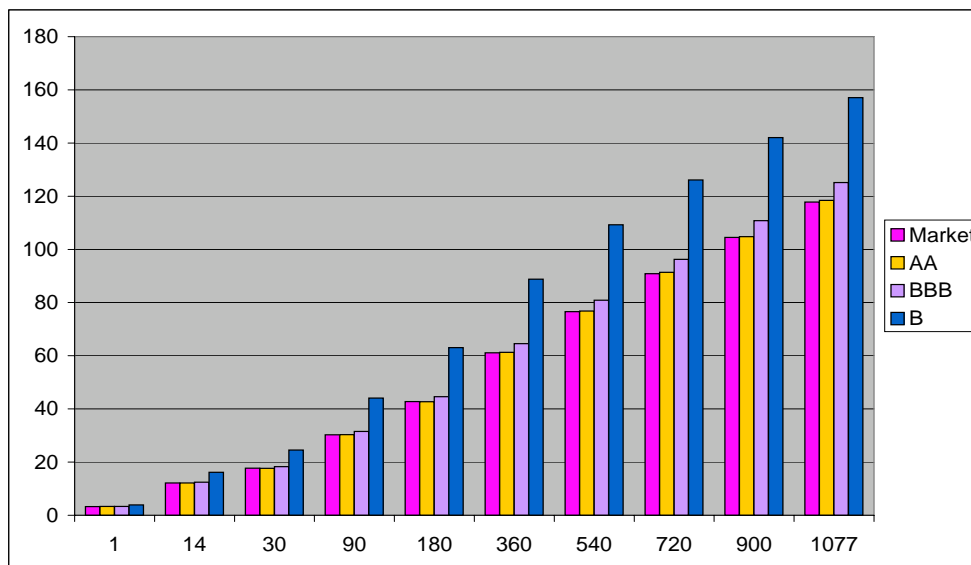


Figure 8: *Stock market VaR and integrated VaR with initial rating AA, BBB and B*

Figure 8 shows the term structures of market VaR, integrated VaRs with initial rating AA, BBB and B at CL 99%. We can see that with the increase of time horizon  $H$ , all the VaRs increase. Among them, the term structures of integrated VaRs for initially AA and BBB rated portfolio follow the term structure of market VaR closely, this is because during the whole period till maturity, credit events rarely happen to high credit quality stock portfolios. In contrast, default events become more likely for low credit quality ones, that can be verified since the integrated VaR for B initially rated portfolio is obviously larger than its market VaR during the holding period.

We have discussed that market VaR would underestimate the risks of the stock portfolios. In Figure 9, the underestimation effects are shown for stock portfolios with initial rating BBB and B. The underestimation for portfolio with initial rating AA is ignored since all ratios of market VaRs VS integrated VaRs are very close to 1 at all time horizons.

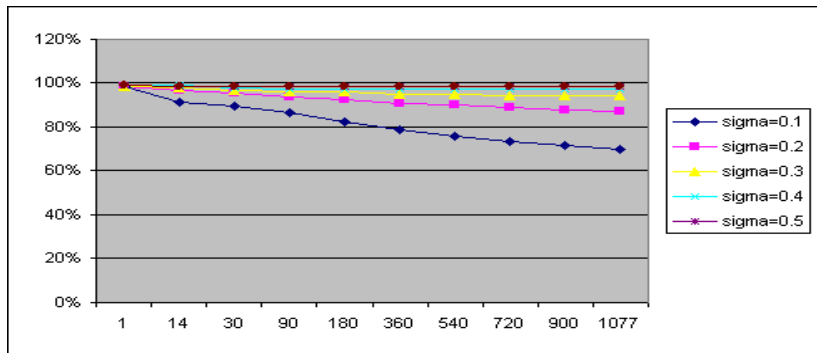
In panel (a) of Figure 9, the underestimation for initially BBB rated stock portfolio is shown. With the increase of  $H$ , all the underestimation effects became more significant, especially for lower stock return volatilities. In panel (b), although serious underestimation also happened to low  $\sigma_s$ , the whole term structures of underestimation for all  $\sigma_s$  become relatively flat concave curves. The reason for lower  $\sigma_s$  corresponding to more serious underestimation is that market VaRs are totally proportional to  $\sigma_s$ , and small  $\sigma_s$  will lead to less dominant market risk compared with credit risk. The shape of the underestimation term structures are actually determined by the relative increase speeds of market risk and credit risk, if market risk increases faster than credit risk, the term structures slope down just like those showed in panel (a), and if market risks increase faster first and then slower than the increase of credit risk, the underestimation term structure would be some concave curve like those in panel (b).

### 3.3.2 Portfolio of stocks and bonds

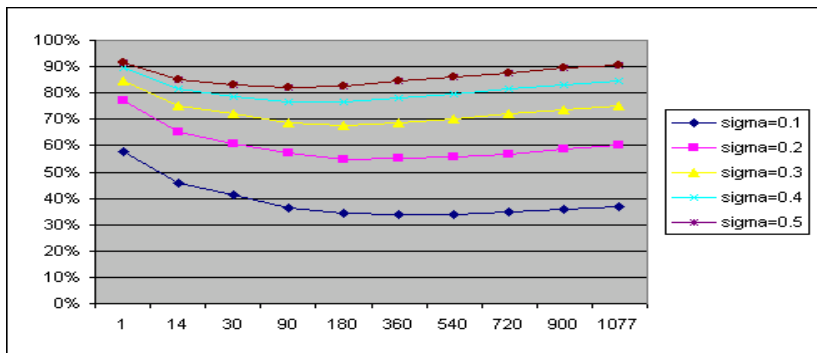
In this section the portfolio consists of 100 exchangeable bonds and 100 exchangeable stocks, which are issued by 200 different companies with equal initial rating and weights. The standardized asset returns for those 200 different companies follow multivariate normal distribution with identical pairwise asset return correlation.

Now we fixed the  $\sigma_s$  at 0.3 and compare the integrated VaRs for portfolios of bonds, portfolio of stocks and portfolio of bonds plus stocks.

Table 8 shows the statistical characteristics for portfolio value distributions of stock portfolio, bond portfolio and portfolio consisting of bonds and stocks, with fixed  $\sigma_s = 0.3$  and  $H=360$  days. As expected, the stds and VaRs for bond portfolios are the smallest,



(a) Stock VaR percentages for portfolio of initial rating BBB



(b) Stock VaR percentages for portfolio of initial rating B

Figure 9: Market VaR VS integrated VaRs for stock portfolio with initial rating BBB and B

Table 8: Comparison of integrated VaRs for different portfolios

AA	mean	std	skew	kurtosis	VaR 95	VaR 99	VaR 99.9
bonds only	214.76	2.450	0.018	3.026	4.0129	5.629	7.598
stocks only	221.05	30.132	0.411	3.298	45.794	61.27	77.593
bonds+stocks	217.86	15.556	0.382	3.265	23.835	31.799	40.332
BBB							
bonds only	214.98	3.038	-0.571	4.940	5.090	8.329	14.313
stocks only	220.61	30.575	0.346	3.311	46.913	64.543	84.201
bonds+stocks	217.8	16.167	0.249	3.317	25.251	35.685	48.451
B							
bonds only	213.548	9.182	-1.136	5.281	17.324	29.183	44.708
stocks only	212.86	36.95	-0.033	3.2152	61.316	88.809	119.99
bonds+stocks	213.18	22.55	-0.290	3.390	38.961	58.579	81.403

while those statistics for stock portfolio are the largest and those for portfolio of stocks plus bonds are in the middle.

Figure 10 shows the integrated VaR term structures for bond portfolio, stock portfolio and portfolio of bonds plus stocks with fixed  $\sigma_s = 0.3$  and CL 99%. The integrated VaRs for portfolio of bonds plus stocks are larger than those of bond portfolio and smaller than those of stock portfolio with all of the three initial ratings - AA, BBB and B. But if the type of assets in the portfolio are determined, the B rated portfolio has the largest VaRs compared with those of AA and BBB rated portfolios. This implies that the type of assets is the most important factor to determine the integrated VaRs, and then the firms' initial credit qualities.

### 3.4 Integrated risk with Non-normal asset return assumptions

In the above calculations and simulations, the asset returns are assumed to follow multivariate normal distribution with mean zero vector and covariance matrix  $\Sigma$ . But in fact asset return distributions are asymmetric and fat-tailed, so in this section, we simulate and calculate the results under assumptions that asset returns follow the fat-tailed multi Student t distribution and asymmetric skew t distribution.

The portfolio consists of 100 exchangeable stocks and 100 exchangeable bonds. To emphasize the effects of different asset return assumptions, H is fixed as one year and the other parameters are kept unchanged.

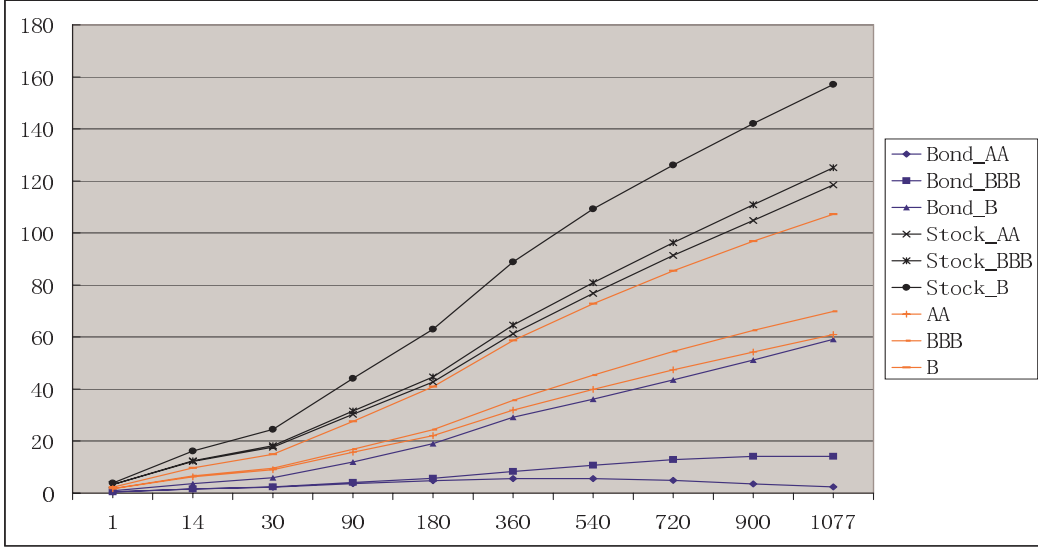


Figure 10: *Integrated VaRs for portfolios of bonds, stocks and bonds+stocks with initial rating AA, BBB and B*

### 3.4.1 Student t distributed asset returns

As shown in Equation (21) in the Methodology section, the multivariate Student t distributed asset returns can be simulated by

$$\mathbf{X}_n^T = \sqrt{\frac{v-2}{v}} \sqrt{W} (\sqrt{\rho_v - \rho_{rv}^2} Z + \rho_{rv} X_r + \sqrt{1 - \rho_v} \varepsilon_n)$$

Which can guarantee that the 200 different firms' asset returns have mean vector zero and covariance matrix  $\Sigma$ . For the bonds, the future ratings at H=1 year are determined by the Student t distributed random variable  $X_n^T$  and the seven thresholds derived from Student t inverse  $t^{-1}$ . In the same way, whether the stocks default or not depends on which one is bigger, the random variable  $X_n^T$  or the threshold of default? If no default happens, the stock prices become  $S_t = S_0 e^{(\mu_s - \frac{1}{2}\sigma_s^2)t + \sigma_s X_n^T \sqrt{t}}$ , note that the  $X_n^G$  in Equation (14) has been changed to  $X_n^T$ .

Now if the degrees of freedom  $v$  is given as 6, then what would the portfolio value distribution looks like under the assumption that the firms asset returns are Student t

distributed?

Table 9: Student t distributed asset returns -Integrated risk

AA	mean	std	skew	kurtosis	VaR 95	VaR 99	VaR 99.9
0.1	217.88	5.4945	0.28595	7.1006	8.3556	13.071	22.139
0.2	217.93	11.056	2.8112	124.53	15.541	23.272	35.36
0.3	218.07	20.159	54.852	13809	22.509	32.334	46.208
0.4	218.39	38.659	159.3	55706	29.228	40.618	55.186
0.5	224.25	3765.7	706.88	5.00E+05	41.352	53.398	68.535
BBB							
0.1	217.66	6.9236	-1.4951	23.002	10.063	22.645	50.742
0.2	217.8	11.93	3.3655	261.5	16.839	29.961	57.392
0.3	218.27	168.03	652.19	4.43E+05	23.785	37.261	62.608
0.4	218.84	283.02	682.79	4.77E+05	30.617	44.652	69.83
0.5	219.58	194.89	450.77	2.53E+05	37.501	51.769	75.532
B							
0.1	211.15	16.904	-1.7438	7.5761	34.187	60.92	91.761
0.2	212.3	19.755	-0.6213	16.737	37.311	63.898	93.983
0.3	213.39	24.593	7.384	919.33	40.587	66.96	95.837
0.4	214.57	43.954	134.91	40192	44.735	70.368	99.833
0.5	228.89	8885.1	706.6	5.00E+05	62.529	87.817	115.79

Table 9 shows the statistical characteristics for portfolio value distribution with Student t distributed firm asset returns. The third, fourth and fifth columns are std, skewness and kurtosis . They are remarkably larger compared to those of normal distributed asset returns, especially the skewness and kurtosis with high  $\sigma_s$ . What's more, they are not very stable for high  $\sigma_s$  and lower credit quality portfolios. One interpretation is that Student t distribution has fatter tails than normal distribution, the effects of extreme events are enlarged dramatically since the std, skewness and kurtosis are in the order of  $(pv - E(pv))^2$ ,  $(pv - E(pv))^3$  and  $(pv - E(pv))^4$  respectively (pv is the abbreviation of 'portfolio Value'). They are not very stable with 500000 simulation paths, but the mean and VaRs at different CLs are relatively stable. The VaRs follow the same trend as that with normal asset return assumption, i.e., they become larger when the initial credit quality of the portfolio worsens and the  $\sigma_s$  becomes higher.

Now the  $\sigma_s$  is fixed at 0.3, the integrated VaRs for this portfolio with different initial ratings and asset return distribution assumptions are calculated and compared. The asset return distributions are Normal, Student t with V=6, 10, 15, 20 and 100 respectively. Ta-

Table 10 summarizes the integrated VaRs, we can see that for all VaRs at CL 99% and 99.9%, the results for portfolio with Student t distributed asset returns are larger than those for portfolio with Normal distributed asset returns, and the difference becomes smaller with the increase of  $v$ . In contrast, VaR95 followed different patterns, except for portfolio with initial rating B, the other VaR95 values for portfolios with Student t distributed asset returns are all smaller than those with normal distributed asset returns, and the difference between them decreases with the increase of  $v$ . These findings are consistent with those of Grundke (2005) among others, and verifies again that Student t distributed asset returns can lead to fatter tails to objective portfolio value distributions, and the distribution will converge to those with normal distributed asset returns if  $v$  goes to infinity.

Table 10: Comparison portfolio VaRs with Normal and Student t distributed asset returns

$\sigma_s = 0.3$	VaR 95	VaR 99	VaR 99.9	VaR 95	VaR 99	VaR 99.9
	Normal			v=6		
AA	23.835	31.799	40.332	22.509	32.275	46.208
BBB	25.251	35.685	48.451	23.785	37.261	62.608
B	38.961	58.579	81.403	40.587	66.96	95.837
	v=10			v=15		
AA	22.97	32.25	43.87	23.207	31.969	42.706
BBB	24.219	36.82	57.414	24.521	36.23	53.915
B	40.023	64.39	93.532	40.078	62.546	89.786
	v=20			v=100		
AA	23.256	31.657	41.947	23.509	31.775	41.443
BBB	25.035	36.77	53.274	25.331	36.226	49.655
B	39.571	61.036	87.841	39.196	59.776	83.86

### 3.4.2 Skew t distributed asset returns

To capture the asymmetric property of asset returns, in this section we model the firm asset returns as multivariate skew t distribution. They can be simulated by following Equation (25)

$$X_n^{st} = \sqrt{\alpha} \left( \mu + W\gamma + \sqrt{W}(\sqrt{\tilde{\rho}_v - \rho_{rv}^2} Z + \rho_{rv} X_r + \sqrt{1 - \tilde{\rho}_v} \varepsilon_n) \right)$$

The future ratings of bonds, the default events and market price changes of stocks are all determined by these skew t distributed asset returns, in the same way described above with normal distributed asset returns.

For skew t distributed asset returns, the skewness parameters  $\gamma$  are usually negative, which has been verified in Hu(2005). Negative  $\gamma$  means left skewed asset return distributions and  $\gamma = 0$  corresponds to Student t distributed asset returns. If the parameters  $v, \rho_v$  and  $\gamma$  are given, the parameters  $\alpha, \mu$  and  $\tilde{\rho}_v$  can be determined from Equation (27).

Table 11 shows the parameters for  $v=6, 15$  and  $100$ . These parameters guarantee that all the nine asset return distributions have the same mean vector **zero** and covariance matrix  $\Sigma$ . For fixed  $v$ , we can see that when the absolute value of negative  $\gamma$  becomes larger, the actual asset return correlation  $\tilde{\rho}_v$  and  $\alpha$  becomes smaller. For fixed  $\gamma$ , the  $\tilde{\rho}_v$  and  $\alpha$  are increase functions of  $v$ .

Table 11: Parameters for skew t distributed asset returns

$\gamma$	$\alpha$	$\mu$	$\tilde{\rho}_v$	$\alpha$	$\mu$	$\tilde{\rho}_v$	$\alpha$	$\mu$	$\tilde{\rho}_v$	
		v=6			v=15			v=100		
-0.1	0.6568	0.15	0.188	0.8469	0.1154	0.1983	0.9798	0.102	0.1998	
-0.2	0.6289	0.3	0.152	0.8595	0.2308	0.1933	0.9792	0.2041	0.1993	
-0.3	0.5874	0.45	0.092	0.8506	0.3462	0.1849	0.9781	0.3061	0.1985	

With these parameters, the integrated VaRs are calculated and compared for portfolio with skew t distributed asset returns with fixed  $v=6$  and varied  $\gamma = -0.1, -0.2, -0.3$ . Table 12 shows these integrated VaR99 values. The columns 2-4 are the VaR 99 data for portfolio with skew t distributed asset return, column 5 and column 6 are those for portfolio with Student t and normal distributed asset returns respectively, which are denoted as  $StVaR$ ,  $TVaR$  and  $NVaR$ . We can see that most of the VaRs satisfy  $StVaR > TVaR > NVaR$ , and the  $StVaRs$  increase with the increase of  $|\gamma|$ . This is caused by the different asset return distribution assumptions, among them, Skew t distributions with larger  $|\gamma|$  have fatter left tails than those with smaller  $|\gamma|$ , T and Normal distributions. The fatter tails then are inherited by the portfolio value distributions.

We also examine the patterns followed by VaR95 and VaR99.9 although the data are not listed here. All the VaR 99.9 satisfy  $StVaR > TVaR > NVaR$ , and  $StVaRs$  are increasing function of  $|\gamma|$ . For the VaR 95, things are totally different. For initial AA and BBB rated portfolios, VaR95 follow the pattern  $StVaR < TVaR < NVaR$  and  $StVaRs$  are decreasing function of  $|\gamma|$ , while for B rated portfolio, the VaR95 follow identical pattern as those with AA and B rated portfolios. One interpretation is that the B rated portfolio value distribution is more left-skewed than others, which is caused by more likely happened default events.

Table 12: Comparison of portfolio VaR99 with Normal , Student t and skew t distributed asset returns

		V=6			VAR 99	
AA	$\gamma = -0.1$	$\gamma = -0.2$	$\gamma = -0.3$	$T$	Normal	
0.1	14.312	15.099	15.776	13.022	12.493	
0.2	24.636	25.828	26.779	23.17	22.402	
0.3	33.508	34.186	34.588	32.363	31.799	
0.4	41.219	41.492	40.891	40.461	40.44	
0.5	47.951	47.311	45.266	48.412	48.21	
BBB						
0.1	25.225	27.971	29.123	22.967	17.894	
0.2	32.521	34.9	37.033	29.713	26.976	
0.3	39.615	41.324	42.86	37.25	35.685	
0.4	45.822	46.903	47.898	43.975	43.931	
0.5	51.717	51.991	50.802	56.309	51.588	
B						
0.1	66.535	70.767	73.986	61.372	47.413	
0.2	68.736	72.58	75.299	64.064	53.397	
0.3	71.051	74.054	76.263	67.172	58.579	
0.4	73.755	75.896	76.515	70.329	64.629	
0.5	76.726	76.991	76.659	75.589	69.428	

Now the  $\sigma_s$  is fixed at 0.3, the  $v$  is taken as 6, 15 and 100, then we compare the integrated VaRs for portfolios with different initial ratings and different asset returns distribution assumptions. The results are shown in Table 13, the Student t and skew t distributions with  $v=100$  are closer to Normal distribution than those with  $v=15$  and  $v=6$ , and the VaR patterns for  $v=15$  and  $v=100$  are almost the same as for those with  $v=6$ , which has already been analyzed above.

Table 13: Comparison of VaRs for Normal ,Student t and Skew t distributed asset returns

$\sigma_s = 0.3$	VaR95	VaR99	VaR99.9	VaR95	VaR99	VaR99.9	VaR95	VaR99	VaR99.9
Normal									
AA	23.835	31.799	40.332						
BBB	25.251	35.685	48.451						
B	38.961	58.579	81.403						
T		v=6			v=15			v=100	
AA	22.444	32.363	46.49	23.207	31.969	42.706	23.509	31.775	41.443
BBB	23.686	37.25	60.984	24.521	36.23	53.915	25.331	36.226	49.655
B	40.34	67.172	96.164	40.078	62.546	89.786	39.196	59.776	83.86
Skew T		v=6 $\gamma = -0.1$			v=15 $\gamma = -0.1$			v=100 $\gamma = -0.1$	
AA	22.254	33.508	52.188	23.09	32.535	44.291	23.641	32.031	41.378
BBB	23.4	39.615	72.657	24.702	37.116	57.249	25.314	36.449	49.95
B	41.897	71.051	102.78	40.027	63.578	92.528	39.308	59.614	83.921
		v=6 $\gamma = -0.2$			v=15 $\gamma = -0.2$			v=100 $\gamma = -0.2$	
AA	21.194	34.186	57.731	23.334	33.619	47.138	23.714	32.281	41.255
BBB	22.384	41.324	83.551	24.989	38.213	59.854	25.352	36.036	49.759
B	42.366	74.054	110.69	41.243	65.92	94.398	39.262	59.702	83.412
		v=6 $\gamma = -0.3$			v=15 $\gamma = -0.3$			v=100 $\gamma = -0.3$	
AA	19.453	34.588	65.72	23.631	33.833	48.729	23.687	32.353	42.29
BBB	20.682	42.86	95.162	25.057	39.947	64.005	25.359	36.39	51.391
B	41.424	76.263	116.68	41.936	67.895	97.305	39.446	60.137	83.664

## 4 Conclusions

In this study, based on the model in Grundke (2005), we make three extensions and establish a new framework, with which both market risk and credit risk can be measured simultaneously.

Within this new framework, the portfolio could consist of various kinds of assets other than only bonds. Although different assets usually have very different risk properties, which lead to the difficulty of integrating their risks synthetically, we have shown that based on the standardized firms' asset return, the integrated VaRs for portfolio of bonds plus stocks could be measured reasonably in a uniform way.

In the industry, the time horizon for market risk is usually 1 day or 10 days while time horizon for credit risk is often chosen to be 1 year. The incompatibility between time horizons for different risks has become one of the barriers to measure the risks simultaneously.

This new framework is extended to be able to deal with continuous time horizons, which means that integrated risks can be measured at any arbitrary time points. In this study, we examine how the different  $H$  affect the risks. We find that with the increase of  $H$ , the term structure of market VaRs for bond portfolio humped in middle, with the maximum market VaRs at about one year while the credit VaRs always increase till maturity. These occur at all confidence levels and all portfolios with different initial credit qualities. In contrast, all the term structures of market VaRs and integrated VaRs are increased for stock portfolios with different initial rating and stock return volatility assumptions. When bonds and stocks are mixed to form a new portfolio, the term structures of integrated VaRs for this portfolio lies between those for bond portfolios and those for stock portfolios. If the type of asset in one portfolio is determined, the initial credit quality plays an important role to integrated VaRs.

In this study, we also examine the overestimation (for add VaR) and underestimation (for dominant VaR) for evaluating the risks of portfolios, and their term structures considering continuous  $H$ . We show that for bond portfolio, the credit VaRs significantly underestimates its risks, especially for portfolio with high initial ratings. The most serious overestimation occurs for portfolios with initial rating AA, BBB and B were at  $H=900$  days, 360 days and 14 days, because at those time points the credit VaR and market VaR are half-half of the add VaR and neither can be ignored. While for stock portfolio, the market VaRs underestimated its risks since market VaRs never include default events, if it happens the stock price would jump to zero. The most significant underestimation occurs to portfolio with lowest  $\sigma_s$  and worst initial credit rating.

Since asset return is the key factor to unify the different assets into one framework, we compare the integrated VaRs for portfolios under different asset return distribution assumptions, which are respectively skew t, student t and Normal distribution and denoted as  $StVaR$ ,  $TVaR$  and  $NVaR$ . We show that with all the distributions sharing common mean vector and covariance matrix, for VaR99 and VaR99.9, the integrated VaRs follow the pattern  $StVaR > TVaR > NVaR$ , and  $StVaRs$  increase with the increase of  $|\gamma|$ , and VaR95 for B initially rated portfolio also followed this pattern. But VaR95 for portfolios with initial rating AA and BBB follow different patterns, i.e.,  $StVaR < TVaR < NVaR$  and VaR95 decrease with the increase of  $|\gamma|$ . This is caused by the fact that among the three distributions, the left skewed skew t distribution has the fattest left tail, which implies that joint defaults are more likely to happen for portfolios under such asset return assumptions than those with the other two assumptions.

With this new framework, we have made three main contributions. First, we can obtain integrated VaRs for objective portfolio at arbitrary time horizons. We can also obtain the VaR term structures and term structures for underestimation and overestimation, which are not possible with previous frameworks. Second, we use the asymmetric

non-normal skew t distribution first to model asset return. As shown in results, the skew t distribution provide more flexible and reliable asset return values, and then more reasonable risk measurements. The third contribution is that, although we illustrate the simulation and calculation procedure with portfolio only consisting of bonds and stocks, this framework also can be easily used to any portfolio consisting of interest rate instruments, credit risk sensitive instruments, equities and foreign exchange products. This is very meaningful to insurers, pension funds and some other financial institutions, since they need to diversify their investments into various kinds of assets.

In this study, we only focus on large homogenous portfolio, and the risk profiles for heterogenous portfolio haven't been explored. Although our portfolio include bonds and stocks, they are exchangeable bonds and exchangeable stocks, with the firms sharing common initial rating and pairwise asset return correlations. It is clear that the heterogeneous idiosyncrasies, such as correlation parameters, bond parameters, stock return and volatilities, and parameters for asset return distributions, if they are not identical, that could cause significant changes to portfolio value distributions and consequently to the integrated VaRs. We can say that with a portfolio consisting of less assets, an extension of the new framework to a heterogeneous portfolio is possible. With this controllable number of assets, we can check specific information and calibrate parameters for these assets one by one, but the computational efforts will still be very intensive. The efforts might be justified if one wants to know how well the large homogenous approximation theory works.

## References

- [1] Barnhill Jr.T.M., and Maxwell W.F.(2002). Modeling Correlated Market and Credit Risk in Fixed Income Portfolios. *Journal of Banking and Finance*,Vol. 26, pp.347-374.
- [2] Bomfim, A.(2002). Counterparty Credit Risk in Interest Rate Swaps during Times of Market Stress. Unpublished manuscript, Federal Reserve Board.
- [3] Basel Committee on Banking Supervision (2003). The New Basel Capital Accord, Basel.
- [4] Black,F. and Scholes, M.J.(1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81. pp. 637-654.
- [5] Bucay,N. and Rosen,D.(1999). Credit risk of an international bond portfolio: A case study. *Alog Research Quaterly*, 2(1), pp. 9-27.
- [6] Credit-Suisse-Financial-Products (1997).CreditRisk+ a Credit Risk Management Framework. Technical Document, available from <http://www.csfb.com/creditrisk>.

- [7] Duffie, D. and Singleton K.J.(1999). Modeling Term Structures of Defaultable Bonds. *Review of Financial Studies*, Vol. 12, No. 4, pp. 687-720.
- [8] Frey, R., and McNeil A.J. (2003). Dependent defaults in models of portfolio credit risk. *Journal of Risk*, Vol. 6, No. 1, pp. 59-92.
- [9] Grundke, P. (2004a). Integrating Interest Rate Risk in Credit Portfolio Models. *Journal of Risk Finance*, Vol. 5, No. 2, pp. 6-15.
- [10] Grundke, P. (2005). Risk Measurement with Integrated Market and Credit Portfolio Models. *Journal of Risk*. Vol. 7, No. 3, pp. 63-94.
- [11] Hu W.B.(2005) Calibration of Multivariate Generalized Hyperbolic Distributions Using the EM Algorithm, with Applications in Risk Management, Portfolio Optimization and Portfolio Credit Risk. The Florida State University. Ph.D dissertation.
- [12] Jarrow, R.A. and Turnbull S.M. (1995). Pricing Derivatives on Financial Securities Subject to Credit Risk. *Journal of Finance*, Vol. 50, No. 1, pp. 53-85.
- [13] Jarrow, R., and Yu F.(2001). Counterparty risk and the pricing of defaultable securities. *Journal of Finance*, 53, 2225C2243.
- [14] Jobst N.J. and Zenios S.A. (2001). Extending credit-risk (pricing) models for simulation and valuation of portfolios of interest rate and credit-risk sensitive securities, Working paper 01-03, HERMES Center on Computational Finance and Economics, University of Cyprus, Nicosia, CY.
- [15] Kiesel, R., Perraudin W. and Taylor A. (2003). The structure of credit risk: spread volatility and ratings transitions. *Journal of Risk*, Vol. 6, No. 1, pp.1-27.
- [16] Kijima, M. and Muromachi Y. (2000). Evaluation of Credit Risk of a Portfolio with Stochastic Interest Rate and Default Processes. *Journal of Risk*, Vol. 3, No. 1, pp. 5-36.
- [17] Li, D. (2001). On default correlation: a Copula function approach, *Journal of Fixed Income*, 9, 43C54.
- [18] Merton,R. (1974). On the Pricing of Corporate Debt: The Risk Structure of Interest Rates, *Journal of Finance*, 29, 449C470.
- [19] Medova, E.A. and Smith, R.G. (2005). A framework to measure integrated risk. *Quantitative Finance*, 5(1): 105-121 2005.
- [20] McNeil A.J., Frey R., and Embrechts P.(2005). Quantitative Risk Management: Concepts, Techniques and Tools. Princeton University Press.

- [21] RiskMetrics-Group (1997). CreditMetricsCTechnical Document, available from <http://www.riskmetrics.com/research>.
- [22] Schobucher, P.J. (2003). Credit Derivatives Pricing Models. John Wiley & Sons, Chichester.
- [23] Tanaka,S. and Muromachi,Y. (2003). An Integrative Risk Evaluation Model for Market Risk and Credit Risk. AFIR Colloquium paper.
- [24] Vasicek, O.A. (1977). An Equilibrium Characterization of the Term Structure. *Journal of Financial Economics*, Vol. 5, No. 2, pp. 177-188.
- [25] Wilde, T.,CreditRisk+. A Credit Risk Management Framework, 1997. Wilson, T.,
- [26] Portfolio credit risk (I). *Risk*, 1997a, 10(9), 111-117.
- [27] Wilson, T., Portfolio credit risk (II). *Risk*, 1997b, 10(10), 56-61.