



# High Performance Stochastic Scenario Generation for Variable Annuity Hedging

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# Outline

- **Variable Annuity Products**
  - ◆ **Types of VA s**
  - ◆ **Withdrawal Benefit Rider**
- **Valuation Discussion**
  - ◆ **Pricing**
  - ◆ **Parameterization**
  - ◆ **Hedging**
- **Case Study**
  - ◆ **Performance**
  - ◆ **Accuracy**
  - ◆ **Issues**
- **Conclusion**



# Variable Annuity Products (1)

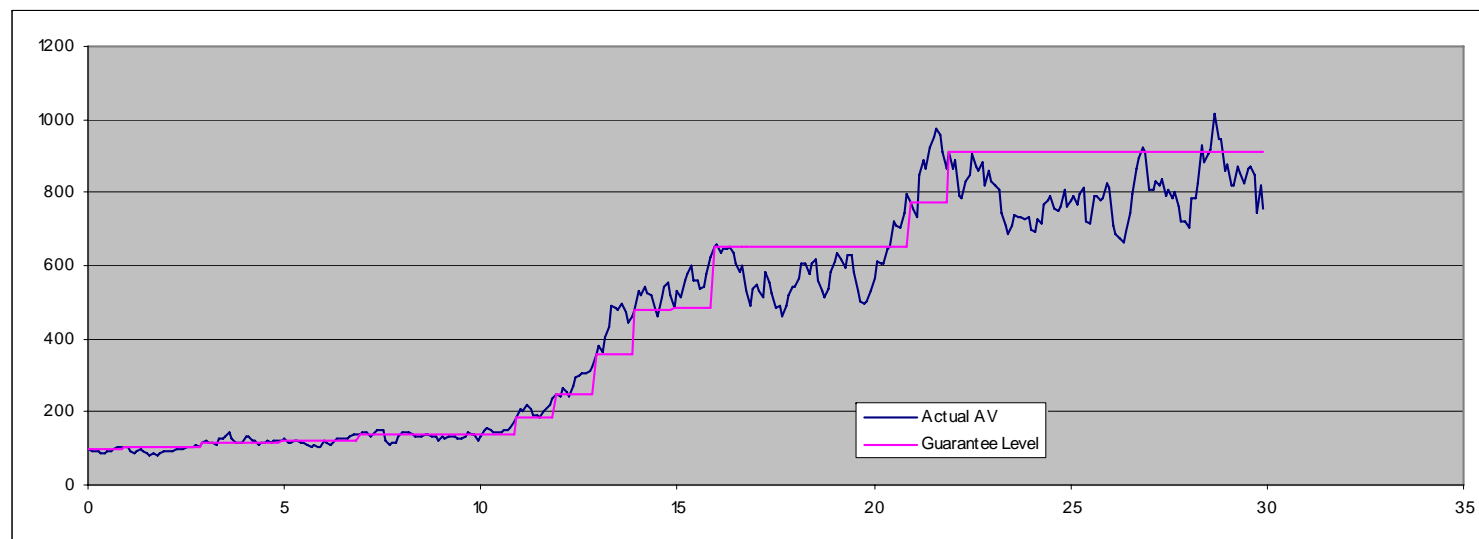
## Types of Variable Annuity Riders

- GMDB – Death benefit clause
- GMAB – Accumulation benefit
- GMIB – Income benefit
- **GMWB – Withdrawal benefit**
  - ◆ Lots of innovation in this space
  - ◆ John Hancock PPFL, IPFL for example
- **Can view WB as a complex financial option**
  - ◆ Complex exercise usually precludes closed form solutions



## Variable Annuity Products (2) Sample GMWB Rider Product

- Underlying account modeled as basket of equities / fixed income elements
- Guarantee has withdrawal features linked to market returns
- Typical withdrawal features include X% of a Guarantee high water mark (e.g. for life) with contractual growth terms as well
- Withdrawal typically begins at age 59.5





# Valuation Discussion (1)

## Valuation and Risks for VA's

- **Monte Carlo Scenario Generator with coupled cash flow generator is common approach**
- **Cashflow generator employs scenario generation results with actuarial assumptions to model payoffs**
  - ◆ **Same engine can handle different “worlds” for scenario generation, e.g. “Real” vs. “Risk Neutral”**
- **Primary hedging concern is risk-neutral world**
  - ◆ **Hedge greeks are calculated via finite difference with repeated operation of cycle**
- **Greeks (Delta, Gamma, Vega, Rho) are used to drive (dynamic and/or static) hedge program**





## Valuation Discussion (2)

### Scenario Generation contexts

- **Scenario Generation used in two distinct modes**
- **Daily Hedging Context**
  - ◆ What to do today
  - ◆ Batch run to calculate updated liability hedge ratios – inner loop “risk neutral”
- **Quarterly Reporting Context**
  - ◆ Reserve and Capital calculation
  - ◆ Capital and Reserve calculations on unhedged basis – outer loop “real world”
  - ◆ Capital and Reserve calculations on hedged basis – outer loop “real world”, inner loop “risk neutral”
    - ❖ The inner loop is our daily hedging context
    - ❖ This “Stochastic-on-Stochastic” mode is very computationally expensive



## Valuation Discussion (3) Parameterization

- **Generally speaking, models with more bells and whistles can explain more variation**
  - ◆ **Examples include volatility skew or stochastic jumps or dividends**
  
- **No Free Lunch: This comes at a cost of additional parameterization**
  - ◆ **How good is my parameter estimation?**
  - ◆ **How good is my calibration?**
  - ◆ **What if I have no data?**
    - ❖ **Even implied correlations can be a problem**



# Valuation Two Worlds

## ■ REAL WORLD

- ◆ Equity Model?
- ◆ Single Economy vs. Multiple Economy?  
Indexes vs Single names?
- ◆ Fund Mapping?
- ◆ RSLN? Stochastic Vol?
- ◆ Interest Rates? Credit Spreads? Term Structure?
- ◆ FX? Inflation?
- ◆ Non traditional (commodity, real estate) investments?
- ◆ Parameterization?
  - ❖ Time steps? Time Horizon?

## ■ RISK NEUTRAL

- ◆ Similar, but requires arbitrage free treatment...
- ◆ Volatility treatment?
  - ❖ Local, Flat, Stochastic, Term Structure?
- ◆ Correlations?
- ◆ Dividend treatment?
  - ❖ Stochastic?  
Deterministic?  
Proportional?  
Constant?
- ◆ Equity jumps?
  - ❖ Arrival parameters, jump size treatment?
- ◆ Interest rate treatment?
  - ❖ Term Structure, Model?
- ◆ Basis risk model?



# Valuation Discussion

## Buy or Build?

- **In house generation**
  - ◆ **Customizable but requires specialized knowledge**
  
- **Off the shelf scenario generators**
  - ◆ **May or may not fit with existing frameworks**
  - ◆ **Does outsourcing parameterization work for your organization?**



# Valuation Discussion

## Hedging with Greeks

- It's all about the greeks
- Performance Attribution uses the Taylor series to explain market profit and losses
- Hedge strategy uses the greeks to offset the first and second order derivatives of the price function
- TSE includes first order and second order terms
- Second order terms are first derivatives of first order terms and can be used as an intraday update for Deltas.

$$\underbrace{f(x + \delta x) - f(x)}_{\text{Change in value of liability}} = \underbrace{\nabla f(x)^T}_{\text{Delta Vector}} \delta x + \frac{1}{2} \delta x^T \underbrace{\nabla^2 f(x)}_{\text{Gamma Matrix}} \delta x + \dots$$

$$\underbrace{\nabla f(x + \delta x)}_{\text{Updated Delta Vector}} = \underbrace{\nabla f(x)}_{\text{Delta Vector}} + \underbrace{\nabla^2 f(x)}_{\text{Gamma Matrix}} \delta x + \dots$$



## Valuation Discussion Hedge Strategy

- **Hedge Strategy and Performance Attribution influence the scenarios that need to be calculated**
- **All hedge strategies address Delta, more sophisticated strategies include Gamma, Rho, Vega**
- **Even if only delta-hedging, Gamma, Rho, Vega, Vomma still useful to explain changes in liability market value**
  - ◆ **Actuarial risk sensitivities are not market hedgeable, but could be measured as well**



# Performance Issues – Daily Hedging (1)

## Case Study

- **Daily Batch Run requires large numbers of scenarios to be generated, stored, processed, post-processed**
- **E.g. consider a basket of 7 equities and a swap curve with 5 key rates**
  - ◆ **Base case run**
  - ◆ **Shocks for first order equity and interest rate greeks**
    - ❖ **E.g. have 12 first order greeks with 24 shocked datasets**
  - ◆ **Second order greeks**
    - ❖ **Gammas can be recycled from central difference delta runs**
    - ❖ **Cross greeks get even more expensive**
      - **E.g. with a basket of 7 equities we have a 12x12 gamma matrix – requires 264 market shocks (=4\*66)**
- **TOTAL:  $1 + 24 + 264 = 2n^2+1$  scenarios (ignoring vega, vomma)**



## Case Study Very Simple Risk Neutral Parameterization

- **Seven index equities, USD, modeled with correlated Geometric Brownian Motions.**
- **Black Scholes ATM 2y vols (vs. vol surface)**
- **Historical correlation matrix (vs. implied correlation!)**
- **Swap curve – 2Y, 5Y, 10Y, 20Y, 30Y key rates**
- **Simulation Horizon 30y with monthly steps**
- **5000 Inner loop scenarios**



# Case Study Greeks Computed

- **7 Delta**
- **5 Rho**
- **49 (28 unique) Gamma**
- **25 (15 unique) IR Gamma**
- **70 (35 unique) Cross Gammas**
- **Totals:**
  - ◆ **12 First order greeks**
  - ◆ **12 “Main Diagonal” Gammas and**
  - ◆ **66 “Off Diagonal” gammas**

$$\begin{bmatrix} \frac{\partial f}{\partial r_1} & \frac{\partial f}{\partial r_2} & \dots & \frac{\partial f}{\partial r_7} \end{bmatrix} \text{ IR Delta ("Rho") Vector}$$

$$\begin{bmatrix} \frac{\partial f}{\partial S_1} & \frac{\partial f}{\partial S_2} & \dots & \frac{\partial f}{\partial S_7} \end{bmatrix} \text{ Equity Delta Vector}$$

$$\begin{bmatrix} \frac{\partial^2 f}{\partial S_1^2} & \frac{\partial^2 f}{\partial S_1 \partial S_2} & \dots & \frac{\partial^2 f}{\partial S_1 \partial S_7} \\ \frac{\partial^2 f}{\partial S_2 \partial S_1} & \frac{\partial^2 f}{\partial S_2^2} & \dots & \frac{\partial^2 f}{\partial S_2 \partial S_7} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial S_7 \partial S_1} & \frac{\partial^2 f}{\partial S_7 \partial S_2} & \dots & \frac{\partial^2 f}{\partial S_7^2} \end{bmatrix} \text{ Equity Gamma Matrix}$$

$$\begin{bmatrix} \frac{\partial^2 f}{\partial r_1^2} & \frac{\partial^2 f}{\partial r_1 \partial r_2} & \dots & \frac{\partial^2 f}{\partial r_1 \partial r_5} \\ \frac{\partial^2 f}{\partial r_2 \partial r_1} & \frac{\partial^2 f}{\partial r_2^2} & \dots & \frac{\partial^2 f}{\partial r_2 \partial r_5} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial r_5 \partial r_1} & \frac{\partial^2 f}{\partial r_5 \partial r_2} & \dots & \frac{\partial^2 f}{\partial r_5^2} \end{bmatrix} \text{ IR Gamma Matrix}$$

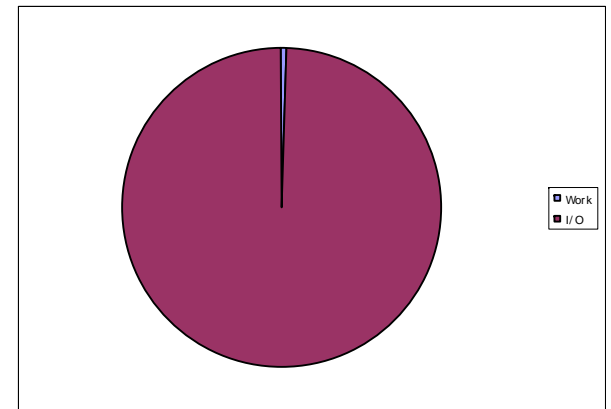
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# Case Study Indicative Results

## ■ Performance

- ◆ 15s to calculate *one* scenario (12.6M rvs)
- ◆ 215s to write to local file (94% I/O)
- ◆ 2645s to write to network file (99% I/O)
- ◆ Comparable time to read from file later in CFG step!



## ■ Accuracy

- ◆ Std error on the 30y Fwd about 2%
- ◆ Using 1000 scenarios, std error jumps to 5%

$$SE_{\bar{x}} = \frac{s}{\sqrt{n}}$$



## Case Study On Accuracy

- **1000 or 5000 Monte Carlo scenarios is not very many in a capital markets context**
- **An ATM call option priced this way has ~10% standard error for the 1000 MC scenarios**
- **Practical issue: variance reduction techniques must be employed**



# Performance Issues

## Summarized Complexity Issues

	Explanatory Sensitivities	Number of Scenario Sets
■ For intraday greek updates as well as performance attribution the number of scenario generations is proportional to the square of n	First order only (Delta, Rho)	$1+2n$
■ Requires fast hardware, and distributed techniques	First order only plus main diagonal gammas	$1+2n$
	Taylor series to second order	$1+2n^2$



# Performance Issues

## Distributed scenario gen. to manage speedup

- **Consider  $1+2n^2$  shocks, 5000 scenarios \* 360 time steps**
  - ◆ **For  $n = 12$ , total work just to create scenarios:**
    - ❖ 1.2h generation of scenarios
    - ❖ 17h to write to local disk
    - ❖ 212h to write to network drive
- **If parallel scenario generation - where do we parallelize?**
  - ◆ **Across shocks?**
  - ◆ **Across scenarios (every machine has all shocks)?**
- **How much scenario return data do we need to store to the network?**
  - ◆ **With basket of 7 equities, market scenarios constitutes 30GB of data**
  - ◆ **Do we really want to be storing this? Is it better to incorporate scenario generation in cashflow generation?**



# Performance Issues Considerations for Distributed Generators

- **RNG cycling**
  - ◆ *Please don't use Ran1 or Excel rand()*
  - ◆ Mersenne Twister is very easy to use, free, etc.
- **Scenario file storage**
  - ◆ Network drives are bottlenecks between Scenario Generation and Cash Flow Generation
  - ◆ Saving scenarios a legitimate audit goal
- **Distribution technique**
  - ◆ E.g. stripe across shocks (load balance issues)



# Closing

- **WB hedging requires a great deal of scenario generation**
  - ◆ For hedge program
  - ◆ For performance attribution
- **Surprising amount of time in I/O**
- **Special techniques must be used to enhance performance of scenario generation**
  - ◆ Variance reduction necessary
  - ◆ Problem is “embarrassingly parallel”
  - ◆ Use of clusters is not optional