Credit Portfolio Optimization under Condition of Multiple Credit Transition Metrics

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Abstract:

Recent years see more and more importance of the management of credit risk for investors, especially institutional investors having large portfolio of corporate bonds, loans or other credit products. Questions like how to evaluate the credit value-at-risk given large amount of information (like different ratings and multiple credit metrics issued by different rating companies), how to build an efficient credit portfolio (having the highest expected return under certain level of credit risk) become increasingly difficult to solve using traditional methods and models. Especially for the second question, the rising dimension of the portfolio under limited computational speed call for leveraging some more robust algorithms for the large portfolio optimization.

In this paper, we will choose JP Morgan's credit metrics model to evaluate the portfolio's credit value-at-risk for the elaboration of our thesis and try to solve the problem of how to leverage multiple credit metrics (as a major input for the model) issued by different rating firms to largely reduce the negative impact of variation of different sources, for the slightest difference among the metrics might result in a huge deviation in the evaluation of the credit risk. At last we will introduce and exploit an increasingly popular and robust algorithm in today's Large Scale Linear Planning Problem---Simulated Annealing to optimize our credit portfolio.

Generally, the paper can be viewed as applied exiting models together with some improving method to better solve today's problem.

Key words:

C-VaR; Credit transition metrics; credit portfolio optimization; Monte Carlo Simulation; Efficient frontier; Simulated Annealing Algorithm

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1 Introduction

In recent years, large attention has been put on credit risk analysis and control not only in academy but also in industry. What is credit risk? Credit risk is the risk of losses that a creditor may have when the obligator cannot pay back all or part of the debt. This kind of risk could exist in bonds, loans, or account receivables. Nowadays, credit risk evaluation models are mainly fall into two groups: single factor models like KMV model based on option theory, JPMorgan's Credit Metrics, CSFP's CreditRisk+ and CreditPortfolioView by Mckinsey as well as multiple factor model like Altman's Z-score, Zeta model, etc.

Multiple factor models are based on financial statements and give rates in a statistical way while single factor models are based on Brown and Poisson process and describe credit risk by simulating Markov process. Both ways of rating risk have their strength and weakness. Financial data is a reflection of the past performance but is not a good indication for the future. Single factor model can well predict the future but too much rely on the credit information provided by rating agency and pay little attention on the market movement as a whole.

In order to alleviate the confliction and negative impact on portfolio decisions brought by multiple rating metrics from different sources, we propose a method that borrows the idea of pessimistic decision to largely reduce the uncertainty. By optimizing the portfolio under this method, that is choose the lowest rating (worst case) as it is and optimize the portfolio given that rating, we can at least secure our position and set us free from worrying about which source of rating is more creditable.

The following paper will cover general introduction a few popular credit risk evaluation models and a step by step demonstration of JP Morgan's CreditMetrics, our method of dealing with multiple Credit Transition Metrics, general introduction of Simulated Annealing Algorithm and a step by step demonstration of our portfolio's optimization process with SA. At last we discuss about the problems what could affect the effectiveness of this method.

2 Popular credit risk evaluation models

2.1 Evolution of the models

Since 1930s, the development of credit risk evaluation models has gone through comparable analysis, statistical analysis and artificial intelligence. In this session we give a brief introduction of the key assumptions and values of various credit risk evaluation models.

- **Comparable analysis in credit risk management**

The traditional credit risk evaluation criteria link credit risk with the default event. The key point is data mining the characteristics of both default and non-default companies establish the identification equations and categorize the samples. The representative model of this stage is 5C analysis, which is Character, Capacity, Capital, Collatera1 and Condition. People try to make a full qualitative analysis about the obligator's willingness and capability of payback from 5 aspects. Early models usually suffer from the subjective, empiricism and lack of objective assessments.

- **Statistical analysis in credit risk management**

After Fisher's research on Heuristics, there developed quickly and enormously credit risk evaluation models based on statistics, of which most represented are Edward·Ahman's Z—score. Edward·Ahman observed manufacturing companies near or far from bankruptcy in 1968 and took 22 financial ratios to establish the most famous 5 varible Z-score based on the mathematical statistical screening. Besides, multiple regression analysis and Logistic regression model, Factor—Logistic model, cluster analysis models and so on. These statistic models' identification functions and the premises of the sample distribution can well interpret the data as well as the coefficients of the model. Yet the weakness lies in the rigidity of the premise such as data should be normal distributed with known variance, which is not easy to find in reality.

- **Artificial intelligence in credit risk management**

With the fast development of information technology, recent years have seen large AI(artificial intelligence) models have been incorporated in the credit risk analysis. For instance, Neural Networks as a self-organizing, self-adapting, and self-learning non-parameter method is very robust and accurate in predicting especially when the distribution is not rigid follow normal.

2.2 CreditMetrics and other single factor models

In this session we will briefly introduce some single factor models which are based on monitoring on the changing process of credit from good to bad and building models on credit rating data. The following are the most famous 4 models developed in the past 2 decades.

- **KMV:**

The KMV model calculates the Expected Default Frequency (EDF) based on the firm capital structure, the volatility of the assets returns and the current asset value. This model best applies to publicly traded companies for which the value of equity is market determined. The translation of the public information into probabilities of default proceeds in 3 stages:

1st Stage: Estimation of the asset value and the volatility of asset return

2nd Stage: Calculation of the distance-to-default
3rd Stage: Derivation of the probabilities of default

- **CreditRisk+:**

Unlike the Merton-based approach and CreditMetrics, the CreditRisk+ methodology is based on mathematical models used in the insurance industry. Instead of absolute levels of default risk – such as 0.25% for a triple B rated issuer – CreditRisk+ models default rates as continuous random variables. Observed default rates for credit ratings vary over time, and the uncertainty in these rates is captured by the default rate volatility estimates (standard deviations). Default correlation is generally caused by external factors such as regional economic strength or industry weakness. CSFP argues that default correlations are difficult to observe and are unstable over time. Instead of trying to model these correlations directly, CreditRisk+ uses the default rate volatilities to capture the effect of default correlations and produce a long tail in the portfolio loss distribution. The minimal data requirements make the model easy to implement, and the analytical calculation of the portfolio loss distribution is very fast.

- **CreditPortfolioView:**

Tom Wilson, formerly of McKinsey, developed a credit portfolio model which takes into account the current macroeconomic environment. Rather than using historical default rate averages calculated from decades of data, CreditPortfolioView uses default probabilities conditional on the current state of the economy. Therefore an obligor rated triple B would have a higher default probability in a recession than in an economic boom. The tabulated portfolio loss distribution is conditioned by the current state of the economy for each country and industry segment.

Here is a table that clearly compares the differences among these popular models and we will introduce JP Morgan's CreditMetrics in detail in the following part by a step by step demonstration of its calculation which our later work largely depend on.

Table 2.1 popular models of credit risk analysis

	CreditMetrics (J.P Morgan)	KMV	CreditRisk+ (CSFP)	CreditPortfolioView (Mckinsey)
MTM (mark to market) or DM (default method)	MTM	MTM or DM	DM	MTM or DM
Source of the risk	Normal distributed returns	Normal distributed returns	Poisson distributed default rate	Macro economic variables
Correlation	Share price and transition probability	Volatility of option and share price	Average default rate correlation	Correlations among various macro economic variables
solution	Algebra or Monte Carlo	Algebra	Algebra	Monte Carlo

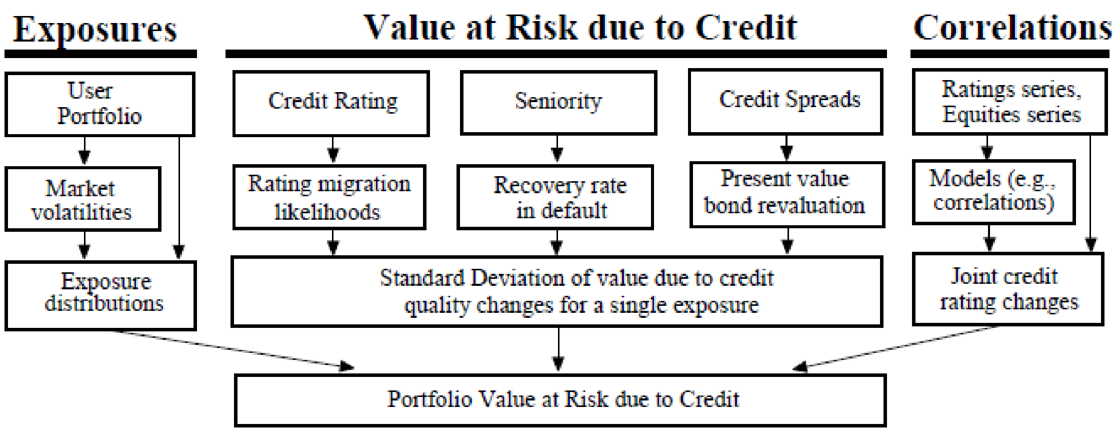
- **CreditMetrics:**

Since our work is largely based on this model and the theories proposed by JP Morgan in 1997, here we give a step by step instruction of how to calculate credit value at risk. We cover the following topics in this instruction part.

- Calculation of the transition probability of different credit ratings and the bond's value at each possible scenario.
- Use of standard deviation as the measurement of credit value at risk of single bond as well as small portfolio.
- Use of Monte Carlo simulation to deal with large portfolio

CreditMetrics is the product for quantifying credit risk developed by JPMorgan in 1997.4, its idea is like Riskmetrics for quantifying market risk published in 1994 in that both are measuring risk by calculating the VaR (Value at Risk)

Here is the logic and framework of CreditMetrics model:



Graph 2.1 CreditMetrics Framework (source: Moody's Carty & Lieberman [96a] and Standard & Poor's Creditweek [15-Apr-96])

.1 Calculation of C-VaR of a 2 bond portfolio :

In this session we will elaborate how to calculate the credit value at risk of a 2 bond portfolio whose composition is like the following:

Senior Unsecured Bond with initial rating of A, 6% coupon and duration of 7 years.

Senior Unsecured Bond with initial rating of B, 5% coupon and duration of 6 years.

We assume at the end of year 1 there are only 3 scenarios: rating will change to A, B and D (default). We can get the value of each bond at a specific scenario using forward interest rate. Detailed calculation and results can be found in the following tables:

Table 2.2 possible values and probabilities at end of year for bond of initial rating A and B

Probability and value of the bond (bond of initial rating A)			Probability and value of the bond (bond of initial rating A)		
Rating at the end of year	probability	value	Rating at the end of year	probability	value
A	0.92	109	A	0.03	108
B	0.7	107	B	0.9	98
D	0.1	51	D	0.07	51

Note: for the bond of initial rating A, the mean and standard deviation are 108.28 and 5.78 respectively; and for the bond of initial rating A, the mean and standard deviation are 95 and 12.19 respectively;

At the end of the year, this portfolio can have 9 different values for 9 different scenarios. For instance, if both bonds remain their initial rating at A and B, the value of the portfolio is 207（=109+98）。

Table 2.3 Possible values of the portfolio at end of the year

Obligator 1 (initial rating A)		Obligator 2 （initial rating B） A	B	D
		108	98	51
A	109	217	207	160
B	107	215	205	158
D	51	159	149	102

Note: the entry of the ith row and the jth column in the middle 3*3 matrix is just the sum of corresponding value at the top row and the left column（for instance, 102=51+51）

.1.1 Joint probability ：

We have already have the independent probability of switching from rating 1(the beginning of the year) to rating 2 (the end of the year) of both bonds, now our problem is to calculate the joint probability of the co-moving of both bonds. Still we simplify it as if the move is independent, thus the joint probability is just the product of the independent probability.

We can derive the joint probability distribution table from table 2.2.

Table 2.4 Joint Probability Distribution

Obligator 1 (initial rating A)		Obligator 2 （initial rating B） A	B	D
	Probability (%)	$p_{21} = p_{B,A} = 3$	$p_{22} = p_{B,B} = 90$	$p_{23} = p_{B,D} = 7$
A	$p_{11} = p_{A,A} = 92$	2.76	82.8	6.44
B	$p_{12} = p_{A,B} = 7$	0.21	6.3	0.49
D	$p_{13} = p_{A,D} = 1$	0.03	0.9	0.07

.1.2 Standard Deviation

Now we demonstrate how to measure C-VaR of portfolio using standard deviation.

$$V_{m,p} = \sum_{i,j=1}^{3} \pi_{ij} V_{i,j} = 203.29 \quad \text{US\$}$$

$$\sigma_{v,p} = \sqrt{\sum_{i,j=1}^{3} (\pi_{ij} V_{i,j}^2) - V_{m,p}^2} = 13.49\text{US\$}$$

If the distribution is normal, we can just use its standard deviation to describe the

$\mathrm{C-VaR} = 1.65\sigma = 1.65 * 13.49 = 22.26$ US$（at 5% significance level）

In sum the C-VaR of corporate bonds depends on the following factors:

1. Joint probability of transition between each risk scenario
2. The portfolio value at each this scenario

Generally speaking, because of the diversification effect on risk, the portfolio C-VaR is smaller than the sum of individual bond.

.1.3 Threshold

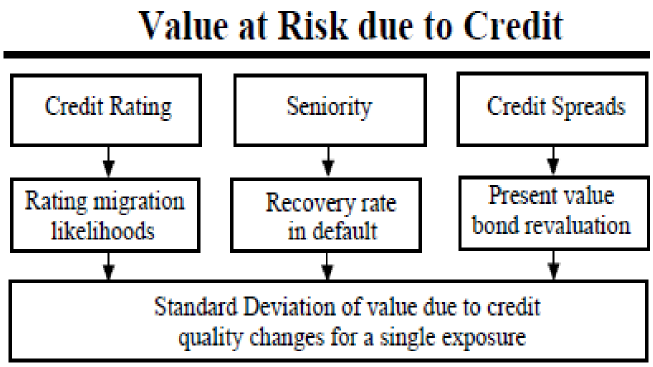
We can use $\sigma_{v,p}$ to roughly describe the portfolio's C-VaR if the distribution is normal while if not, it would be better to use a preset threshold say 1% of the total value distribution. We must first sort the whole possible values and sum the probability of them from the smallest one until we got 1% accumulating probability. For example:

V_{A+B} = {102 US $, 149 US $, 158 US $, 159 US $, …, 217 US $,}

π_{ij} = {0.07, 0.9, 0.49, 0.03, …, 2.76}

Hence, the nearest value when accumulating probability is 1% is 149 US $，and the C-VaR is therefore 54.29 US $，（$= V_{m,p} - 149\mathrm{US\$} = 203.29\ \mathrm{US\$} - 149\ \ \mathrm{US\$}$）

In following graph, we summarize some key points when to calculate C-VaR.



Graph 2.2 Key steps in calculating the C-VaR

.2 Calculation of C-VaR of a 2 bond portfolio ：

In reality, it is more common to deal with large portfolio with N assets. Unfortunately, we can't get the joint transition probability matrix easily needless to say the following standard deviation calculation. Think of a bank with 30 debt portfolio and having 8^{30} joint transition probabilities. In order to better describe the situation and solve the problem of lack of historical data, CreditMetrics tend to use Monte Carlo simulation to get the joint transition probability distribution and what we need is simply the return distribution on the stork market. Steps in this process include：

1. Get asset return threshold Z matrix for each obligator

2. Automatically Generate a series of asset returns following multiple normal distribution
3. Get the credit transition information for each asset return by looking for the asset return threshold Z matrix (for instance, company 1 switch from BBB to CCC) and re-calculate the portfolio value at each new rating to get $V_P^{(1)}$ 。
4. Repeat step 3, for all the simulated return portfolio, we can get a series of $V_P^{(i)}$. We sort them in ascending order and establish a 1% threshold to get the C-VaR.

In the following of optimizing portfolio with multiple assets, we will use this way of calculating C-VaR and demonstrate it in detail.

3 Our data selection and processing

In this session we first interpret the data we use and will do some basic processing of the data for the latter work. Considering the constraint of our resource and capability, we only establish a visual portfolio with 6 bonds. The source of all the information is from Yahoo Finance on 2007-Mar-21.

3.1 Basic information of the portfolio

In our portfolio, the initial investment is 10,000 US $ allocating on 6 senior secured bonds which are all due to March, 2011 and having principal value of 100 US $. Ratings and coupon information are as follows：

Table 3.1 **Rating and Coupon**

	Coupon	S&P	Moody	Fitch
MERRILL LYNCH	7.00%	AA	AA	AA
WALMART	3.38%	AA	AA	AA
BOEING	5.80%	A	A	A
COLA	5.75%	A	A	A
3M	4.20%	BBB	BBB	BBB
TIME WARNER	7.48%	BBB	BBB	BBB

Source：http://finance.yahoo.com/bonds

3.2 Rate of return in stock market

Meanwhile, in order to link to the joint transition probability distribution we need the rate of return of the stock market for these bonds' issuing companies. Hence we collect the annual return of the stock listed on NYSE from 1996-Jun-14 to 2006-Jun-14.

Table 3.2 **Annual return of the stock s of the bond issuing companies**

	MERRILL LYNCH	WALMART	BOEING	COLA	3M	TIME WARNER
2006	16.90%	-4.05%	26.48%	-16.20%	4.14%	1.07%

2005	-0.95%	-12.65%	25.37%	-23.60%	-10.50%	-2.99%
2004	16.88%	4.17%	32.33%	34.99%	-40.61%	10.66%
2003	18.89%	-5.17%	-18.52%	-14.71%	1.38%	-5.32%
2002	-44.55%	14.02%	-40.18%	33.66%	4.29%	-112.29%
2001	-62.55%	-10.77%	49.33%	-6.21%	36.07%	-3.95%
2000	54.70%	24.55%	-7.45%	-70.37%	-5.84%	-54.57%
1999	-22.76%	-27.11%	-3.96%	-11.71%	9.70%	6.13%
1998	34.37%	56.96%	-27.69%	52.81%	-19.81%	37.50%
1997	-1.96%	20.30%	-37.36%	-40.05%	36.77%	26.24%

3.3 Yield curve

In order to get the portfolio value at the end of the year, we need to first calculate the individual bond value by means of the discounting all the remaining coupons and possible value at expiration using the forward rate given specific possible rating. For an AAA rating bond there is a couple of forward rates (we can get them from calculation of its spot rate). These forward rates can be viewed as the best prediction of the future spot rate by the market.

Table 3.3 **Average forward rate of various rating bonds expiration at March, 2011**

	Forward Interest Rate (%)			
	F12	F13	F14	F15
AAA	4.1500	4.2340	4.2060	4.1960
AA	5.6740	6.2080	6.2250	6.2440
A	6.2480	6.8990	7.4630	7.0260
BBB	6.5220	7.1890	7.2400	7.7780
BB	7.2910	7.5550	7.6890	8.8940
B	8.7260	8.7360	9.7590	10.2200
CCC	9.7360	10.4570	11.1950	12.2810

F12 means the interest rate from the end of year 1 to the end of year 2

4 Optimization of credit portfolio with multiple assets

As for large portfolio, it is not easy to calculate the joint transition probability matrix. Think about N obligators with 8 possible rating scenarios. There would be 8^N possibilities! Hence, we use Monte Carlo simulation to roughly generate the value distribution of the portfolio instead of calculating every entry of the matrix. All we need for data input is just the return of the stock. In this session, we will make a step by step demonstration of the whole process. In the optimization, we introduce our proposed pessimistic decision method to deal with the multiple transition matrixes as the data input to reduce the negative effect of the uncertainties. Later, we will also leverage a very efficient and powerful algorithm--- Simulated Annealing to get the most optimized weight of the individual assets in the portfolio.

4.1 Calculation of credit value at risk

4.1.1 Possible values of the individual bond at the end of the year

Here, we calculate the value of each bond at the end of the year using the famous bond pricing formula which we mentioned in the previous session::

$$V_{X,A} = \sum_{n=1}^{n} \frac{C_X}{(1+F_{A(1,n)})^{n-1}} + \frac{C_X + 100}{(1+F_{A(1,5)})^{n}}$$

$V_{X,A}$ is the market value of bond X at rating A by the end of the year

C_X is the coupon of the bond

$F_{A(1,n)}$ is the forward rate of average A –rating bond from the end of year 1 to the end of year n

Given the yield curve we use in table 3.3, the value of the bond issued by Merrill Lynch at the end of the year when its rating switch to A –rating would be:

$$V_{ML,A} = \sum_{n=1}^{4} \frac{C_{ML}}{(1+F_{A(1,n)})^{n-1}} + \frac{C_{ML} + 100}{(1+F_{A(1,5)})^{4}}$$

$$= 7 + \frac{7}{(1+0.06248)^1} + \frac{7}{(1+0.06899)^2} + \frac{7}{(1+0.07463)^3} + \frac{7+100}{(1+0.07026)^4}$$

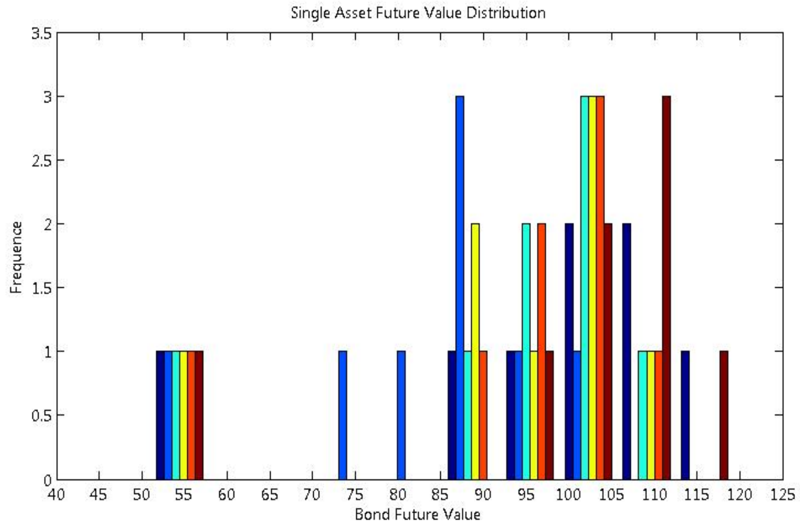
$$= 106.9095$$

With the same way, we can get Merrill Lynch and other 5 bonds' market value when their rating switching to other possible ratings. Of course, even at default event, investors still can get part of the investment. This is determined by the seniority of the bond. Here we have 6 senior unsecured bonds; hence the average recover rate is 51%, which means we can get 51% of the principal when it defaults.

The following table is the market value of individual bonds given ratings:

Table 4.1 **market value of individual bonds at given ratings**

	MERRILL LYNCH	WALMART	BOEING	COLA	3M	TIME WARNER
AAA	117.13	100.41	111.59	111.36	111.59	119.34
AA	109.65	93.51	104.31	104.08	104.31	111.78
A	106.91	91.01	101.64	101.42	101.64	109.01
BBB	104.64	88.83	99.40	99.19	99.40	106.73
BB	101.28	85.66	96.11	95.89	96.11	103.35
B	97.15	81.93	92.11	91.90	92.11	99.17
CCC	91.53	76.71	86.63	86.42	86.63	93.49
D	51.00	51.00	51.00	51.00	51.00	51.00



Graph 4.1 market value distribution of individual bonds at given ratings

Up till now we only get the individual value and the portfolio value is determined by:

① The weight of the individual asset in the portfolio

② Joint credit rating transitions of the portfolio assets

We will cover that in the following part.

4.1.2 Asset return threshold Z matrix

We assume here that the movement in the stock market reflect the change in credit ratings to some extend, hence can be used as a signal of the credit rating change.

First, we need the annual transition metrics for corporate bonds issued by S&P, Moody and Fitch. The following table is annual corporate credit transition metrics issued by S&P.

Table 4.2 S&P 1 year corporate bond credit transition metrics

Average Annual Global Corporate Transition Matrix 1983-2002								
	AAA	AA	A	BBB	BB	B	CCC	D
AAA	96.54	3.31	0.14	0.01	0.00	0.00	0.00	0.00
AA	0.09	90.99	8.47	0.40	0.03	0.02	0.00	0.00
A	0.03	2.50	91.78	5.28	0.24	0.02	0.10	0.05
BBB	0.00	0.25	4.85	89.26	3.97	0.87	0.40	0.40
BB	0.07	0.13	0.20	7.33	79.39	8.06	2.71	2.11
B	0.00	0.00	0.00	0.51	8.08	83.83	5.01	2.57
CCC	0.00	0.00	0.00	0.44	0.00	10.62	58.85	30.09

Source：Roberto Violi；Credit Ratings Transition in Structured Finance (+)；CGFS Working Group on Ratings in Structured Finance

From the above table, we can tell that a BBB rating bond has 0.40% chances of default at the end of the year. Then, if we assume the rate of return on the stock market is normally distributed, we can establish a one to one relationship link the stock return with the company’s credit rating as follows:

$\Pr(\text{default}) = \Pr\{R \prec Z_{Def}\} = \Phi(Z_{Def} / \sigma) = 0.40\%$

Pr is the probability of a specific change in credit rating，R is the annual rate of return on the stock（in order to simplifies our calculation, we standardized the return to make it has zero expected return）.$\Phi(\ )$ is a standard normal cumulative distribution function. σ is the volatility of the share price of a company with initial rating of BBB. We can use this equation to get Z_{Def}， $Z_{Def} = \Phi^{-1}(0.40\%)\sigma = -2.9677\sigma$

To make this reverse process, we use Matlab and get the result immediately $Z_{Def} = -2.9677\sigma$。 Hence, in other words, if we find the rate of return of stock issued by the same company which issued a initial BBB corporate bond decreased by larger than -2.9677σ, this bond may downgrade to default. We can keep calculating other threshold rate of return Z at which the bond credit rating may change. For instance, we observed that this BBB bond has 0.40% chances to become a CCC bond after one year, then:

$\Pr(CCC) = \Pr\{Z_{Def} \prec R \prec Z_{CCC}\} = \Phi(Z_{CCC} / \sigma) - \Phi(Z_{Def} / \sigma) = 0.40\%$

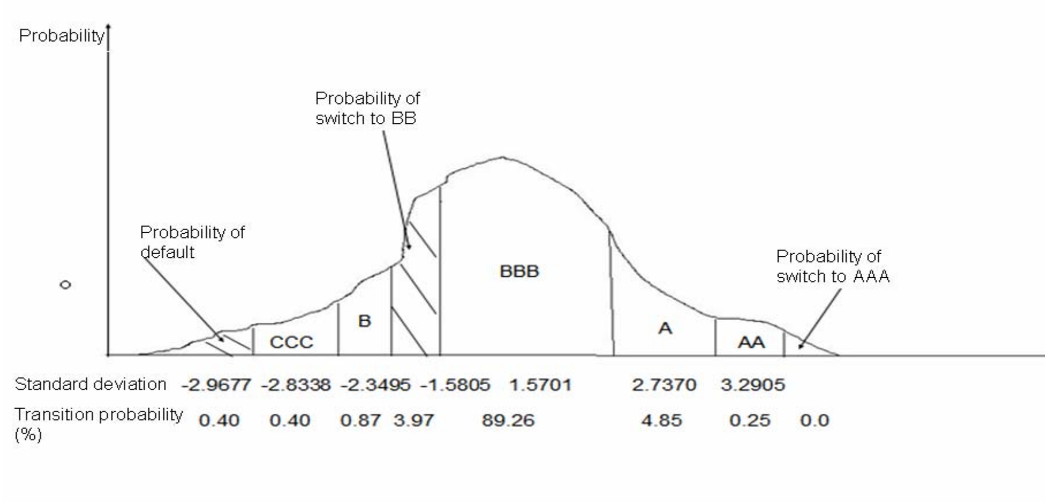
$\therefore \Phi(Z_{CCC} / \sigma) = 0.004 + \Phi(Z_{Def} / \sigma) = 0.008$

$\therefore Z_{CCC} = \Phi^{-1}(0.008)\sigma = -2.8338\sigma$

Further, we can consider eliminate this σ（asset return's volatility)，because the joint credit transition probability doesn't determined by this σ. Think about two obligators with the same credit ratings yet have different variance of the asset return. Assume they are following this equation $\sigma^* = 2\sigma$. This means the volatility of one of the obligator's is twice that of the other. This also means the asset return threshold Z will be twice of the other. Hence, the probability of transition between different ratings will never change with different σ. Now, we can use the standardized normal distribution of rate of return which follows N（0，1）.

Hence, $Z_{CCC} = \Phi^{-1}(0.008)\sigma = -2.8338\sigma = -2.8338$

Similarly, we can get all the threshold asset return Z which triggers rating to switch to other grades as follows:



Graph 4.2 Asset return threshold Z and the credit transition probability of the initial BBB bond

Repeat the above steps, for all bonds with different initial ratings like A, AA, A……CCC, we can get an asset return threshold Z matrix for the credit transition probability metrics issued by S&P as follows:

Table 4.3 S&P Asset Return Threshold Z Matrix

	AAA	AA	A	BBB	BB	B	CCC
AAA	—	—	—	—	—	—	—
AA	-1.506	2.142	3.195	3.291	3.540	8.210	—
A	-2.478	-1.416	1.986	2.737	3.195	3.353	—
BBB	-3.540	-2.669	-1.625	1.570	2.583	2.929	—
BB	-3.540	-3.090	-2.524	-1.581	1.585	2.452	2.495
B	—	-3.719	-3.012	-2.350	-1.444	1.471	1.930
CCC	—	—	-3.719	-2.834	-2.162	-1.358	1.495
D	—	—	—	-2.968	-2.229	-1.490	-0.704

By the same way we can also get this matrix based on the credit transition probability metrics issued by Moody and Fitch.

4.1.3 Simulate rate of asset return by Monte Carlo

In this session we use Monte Carlo simulation to approximately get the credit portfolio value distribution using the historical data from stock market and the linkage between stock market and credit rating transition. By this way, we no longer suffer from the dimension problem mentioned previously (think about that $8^6 = 262144$)! Besides, we only have 10 year's observations and lack the extreme scenario. Monte Carlo simulation perfectly saves us a lot of time for calculation as well as providing enough data entries.

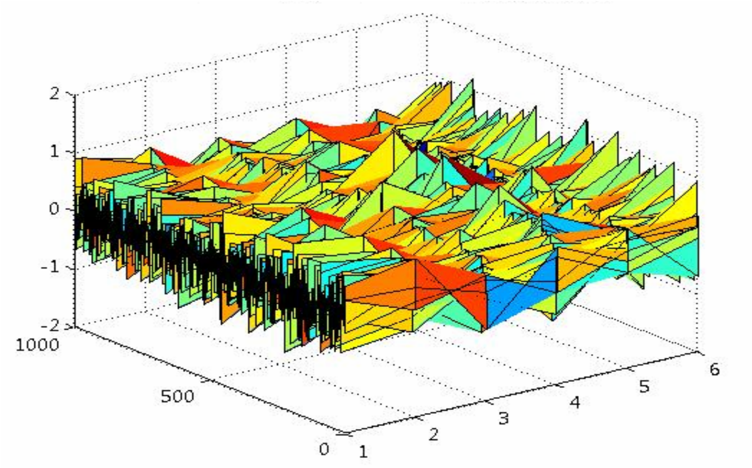
- **Key steps of Monte Carlo Simulation**

1. Construct and describe the distribution：

We assume the stock return is normally distributed. Hence, in our Monte Carlo simulation, we use the mean vector as well as covariance of the historical data of those 6 companies we pick as the input variables.

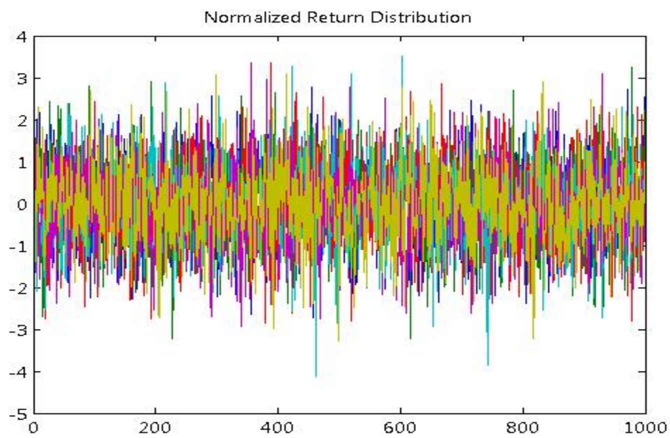
2. Generate a sample from the establishing distribution:

After we construct our distribution, the key of the Monte Carlo simulation is to generate random variables as our sample from that distribution. We use the random generator in Matlab to generate 1000 vectors which follow the distribution we previous set. Each vector consist 6 possible asset return.



Graph 4.3 Multiple normal distribution of stock returns by Monte Carlo Simulation

Again, in order to simplify our later work, we standardized our return by equation $X = (X - \mu)/\sigma$ and get the standardized normal distribution of return：



Graph 4.4 standardized normal distribution of return

4.1.4 Determine a possible credit transition

Last session, we have generated 1000 simulated return vectors and（1.4741, 2.7361, -2.5047, 1.0696, -0.82376, 0.63461）is one of them. We will calculate its credit transition status after one year. These 6 data point are the stock returns of those 6 companies issuing the bonds. For instance, $R_{ML} = 1.4741$ is the possible return (standardized) of Merrill Lynch. Moreover, the initial of credit ratings of these 6 bonds are（AA, AA, A, A, BBB, BBB）.

From the Asset Return Threshold Z Matrix we get from the previous session, we can find the triggering point Z that cause the initial AA bond's credit rating switching to other grades. Since $Z_A \prec R_{ML} = 1.4741 \prec Z_{AA}$， we are confident that the Merrill Lynch's bond will remain the same rating at the end of the year.

By the same way, we can get the possible credit ratings for the other 5 bonds after one year. Then we get a credit transition vector after one year like（AA, AAA, BBB, A, BBB, BBB）. Comparing with the initial one (AA, AA, A, A, BBB, BBB）, we find that Wal-Mart's bond upgraded by one unit and Boeing's downgraded by one unit with all the others remain the same. That is because the change in the stock return of Wal-Mart and Boeing excess the triggering point get from asset return threshold Z matrix. Now, we have established a one to one relation between stock return volatility and credit rating transition.

4.1.5 Revaluate the portfolio under new credit rating

According to the individual bond value under different credit rating transition from the initial one, which we calculate in the very beginning of this part, we can immediately get the value of Merrill Lynch's bond with AA rating, which is $V_{ML,AA} = 109.65$。

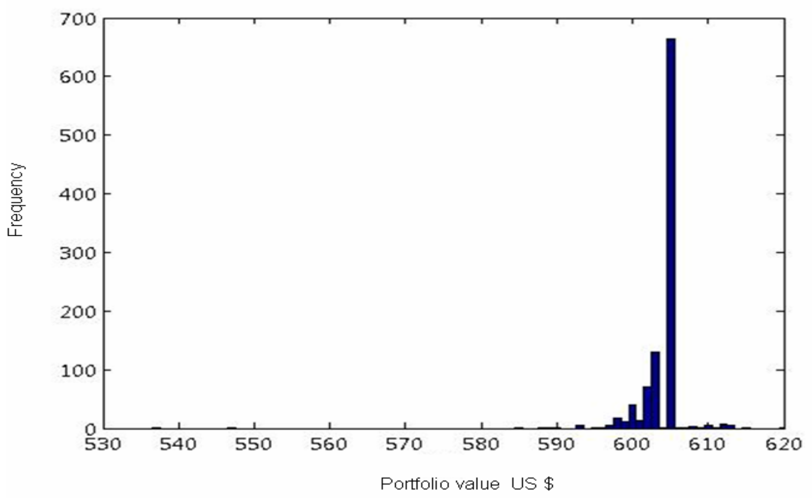
By the same way, we can get the market value of other 5 bonds after the credit transition. The new credit rating is（AA, AAA, BBB, A, BBB, BBB）, therefore the new market value vector is（109.65, 100.41, 99.40, 101.42, 99.40, 106.73）.

For the portfolio, its total value depends not only on the individual values of the bonds but also their weights. Different weight of combination will affect the overall expected return for the portfolio as well as the credit risk. Therefore, we put lots of effort on how to efficiently and effectively optimize the portfolio given the specific risk preference. Since we will talk about this later and here for demonstration convenience we only assume the weights of all 6 bonds in the portfolio are the same. Thus, the portfolio value is:

$$V_P = 109.65 + 100.41 + 99.40 + 101.42 + 99.40 + 106.73 = 617.01$$

4.1.6 Portfolio value distribution and C-VaR at 5% significance level

We have 1000 possible return combinations and by repeating the revaluating process like in the previous session we can get a distribution of the portfolio value at the end of the year as follows:



Graph 4.5 Credit Portfolio value distribution at the end of year (by Monte Carlo simulation)

We sort the above 1000 values in an ascending order and at the top 5%, we have our threshold value of 599.43 US $ which is the market value of the 50th data (5%*1000=50). Hence, the portfolio credit value at risk is simply that value at top 5% minus the sample mean: $\text{C-VaR} = V_{P,5\%} - \overline{V_P} = 599.4336 - 604.1141 = -4.6805$

It tells us that after 1 year, there is 5% chance that the loss of this portfolio because of the credit transition will exceed 4.6805 US $.

Up till now, we have finished the portfolio credit value at risk calculation part, which covers the following key steps:

1. Look at individual obligator's stock and credit ratings transition; get the asset return threshold Z matrix.
2. Generate a series of return vectors by Monte Carlo Simulation
3. Determine a new credit rating for each return vector and revaluate the portfolio value $V_P^{(1)}$。
4. Repeat step 3， for all the simulated return combination, we get a series of $V_P^{(i)}$. Sort them in an ascending order. Calculate the credit value at risk at a specific significance level, say 5%

In the following part, we will mainly focus on how to optimize the portfolio. Minimizing the credit value at risk for a given return, as one of the optimization objectives depends on the calculation of C-VaR, which basically based on our former work.

4.2 Credit Portfolio Optimization---our method and techniques

Our objective is to find an optimal weight of the individual assets that maximizes the portfolio expected return meanwhile minimizes its credit risk. This is the principal of the following work and our job is to try to find an easy and efficient way to solve the following two big problems.

1. How to value the goodness of multiple credit rating information from different rating companies?

Considering only 3 independent rating firms: S&P，Moody and Fitch. These organizations are independent from each other, issuing ratings for companies and industries based on their own source of information and credit risk models. Even though the results seem not differ far from each other, there is still variation and sometimes this small difference will bring huge negative effect for investment decisions. For example, in our experiments, we find out that comparing with S&P and Moody，Fitch seems more "strict", for the issued data are most conservative. This may result from its source of information observed or simply because of its more strict models. Whatever, we try to alleviate this negative affect because of the difference ratings for the same company.

2. It always subject to the time and space constraint to find out the optimal portfolio.

In our portfolio, the initial investment is 10000 US $ allocating among 6 bonds (in real word, the size of portfolio may much larger). The principal of bond is 100 US $, therefore the problem is simply finding the optimal weights and these weights sum up to 100（10000/100）. Actually, it is much more complex than it looks like. If we don't apply any algorithm, just let the computer try every possible combination, the efficiency is $O(m^t O(u*n))$, where,

m is the sum of all weights
t is the number of assets

$O(u*n)$ is the time need to calculate the C-VaR for one specific combination, where,

u is the independent rating firms
n is the random sample generated by Monte Carlo Simulation

Take our experiment as an example, assume the processing power of our computer is 1 M flops, therefore the time needed is $t = 100^6 * 15/10^6 = 1.5*10^7 s \approx 173.6 days$ （from where we already know the time for calculating the C-VaR for one specific combination is $O(3*1000)$, approximately 15 seconds）

Hence, the goodness of algorithm significantly affects the processing power of the optimization problem in portfolio investment and refrain the size of the portfolio.

Therefore, we use the following method to solve the above two problems.

1. We propose a method that borrows the idea from pessimistic decision to largely reduce the negative effect by uncertainty of the credibility on multiple rating information.
2. We incorporate a very robust algorithm, Simulated Annealing to optimize the portfolio.

4.2.1 Pessimistic decision

When there are more than two scenarios and the probability of them can't be confirmed, we can call this decision problem uncertainty decision. This decision problem has complex constraints and large set of variables most of which can't be quantified; hence it's not an easy job to establish the mathematical models. What's more, the variables and correlations among them are uncertain, therefore making it impossible to build the objective function to get the optimal solution.

Popular methods for uncertainty decision problem are as follows:

- Maximize the minimum return
- Minimize the maximum regret
- Maximize the maximum return
- Optimistic coefficient

In our work, we develop that Maximize the minimum return method and get our pessimistic decision method. Follows is the comparison of the two methods.

- Maximize the minimum return

  This method need to first calculate all possible return under each option and find the option with the maximum return as the optimal solution. This is a more conservative way of decision making, from the perspective of the worst cases.

- Our pessimistic decision (Minimize the maximum loss)

We first find the largest credit value at risk under all credit rating firms' transition metrics. Next, we try to find the optimal weight that minimize this maximum C-VaR. We stand on the point of the worst case and take this as our optimization objective. By this way, investors can avoid the negative effect by incautiously choose one rating firm or simply make a weighted average of all the sources.

4.2.2 Simulated Annealing

As its name implies, the Simulated Annealing (SA) exploits an analogy between the way in which a metal cools and freezes into a minimum energy crystalline structure (the annealing process) and the search for a minimum in a more general system.

The algorithm is based upon that of Metropolis, which was originally proposed as a means of finding the equilibrium configuration of a collection of atoms at a given temperature. The connection between this algorithm and mathematical minimization was first noted by Pincus, but it was Kirkpatrick who proposed that it form the basis of an optimization technique for combinatorial (and other) problems.

Simulated Annealing (SA)'s major advantage over other methods is an ability to avoid becoming trapped at local minima. The algorithm employs a random search which not only accepts changes that decrease objective function, but also some changes that increase it. The latter are accepted with a probability.

- **Simulated Annealing –the model**

Simulated Annealing Algorithm consists of 3 parts, solution space, objective function, and initial solution.

1. Solution space：

It is the group of all possible solution and it restrict the scope of our choosing the initial solution and the new solution. In many optimization problems, besides objective functions, we also have a set of constraints. Hence, there might be some infeasible so1ution in the solution space. You can define the solution space exclusive of infeasible so1utions or you can allow them by incorporating a penalty function to penalize the occurrence of the infeasible so1ution.

2. Objective function：

It is the mathematical description of the optimization problem. Usually it is constructed as the sum of several optimization targets. The choice of objective function should well reflect the optimization requirement and as mentioned above, when infeasible so1ution are allowed, objective function need to incorporate a penalty function.

3. Initial solution：

It is the starting point of the algorithm. It has been proved that Simulated Annealing Algorithm is robust, and the final solution is independent of the choice of the initial solution.

- **Simulated Annealing –the idea:**
  - ✓ Initialize：making initial temperature large enough, setting initial solution S(the starting point of the itinerary of the algorithm)， L（Markov chain length）for the times of itinerary for each temperature value T.
  - ✓ For k=1，……，L do step 3 to step 6
  - ✓ Generate new solution S′
  - ✓ Calculate the incremental Δt′=C(S′)-C(S)，where C(S) is the comment function
  - ✓ Find the transition probability P_t of solution according to Metropolis principal. Decide whether to accept the new solution or not.
  - ✓ $P_t(i \Rightarrow j) = \begin{cases} 1, & \text{when } f(j) \le f(i) \\ \exp(\frac{f(i)-f(j)}{t}), & \end{cases}$
  - ✓ If the stop condition has been satisfied, we replace the current solution as the optimal one and cease the program.
  - ✓ T declines gradually and T>0，return to step 2
- **Simulated Annealing –the Pseudocode**

procedure SIMULATED-ANNEALING;

begin

INITIALIZE (i_0, t_0, L_0);

k:=0;

i:= i_0 ;

repeat

for l: =1 to L_k do

begin

GENERATE (j from S_i);

if f(j) >= f(i) then i:=j

else

if exp($\frac{(f(i) - f(j))}{t_k}$) > random[0,1) then i:=j

end

end

k:=k+1;

CALCULATE-LENGTH (L_k):

CALCULATE-CONTROL (t_k)

until stopcriterion

✓ end;

4.2.3 Our optimization target and constraints

Our objective is to find an optimal weight of the individual assets that maximizes the portfolio expected return meanwhile minimizes its credit risk. As mentioned above we first find the largest C-VaR, then find the optimal weights to minimize this maximum C-VaR. Since for a give risk preference there will be a corresponding optimal weights to minimize the credit risk as well as maximize the expected value of the portfolio, our output will be a series of points consist of an effective frontier line. Our objective function and constraints are as follows:

$$\begin{cases} \mathrm{Max}_X \{ \mathrm{Min}_m (CV_m(V_P) + \tau * \mu_m(V_P)) \} & m = 1,2,3 \\ V_P = \sum_{i=1}^{6} (X_i V_i) & \\ \sum_{i=1}^{6} X_i = 100 & \\ 95 \succ X_i \succ 0, X_i \in N & i = 1 \sim 6 \end{cases}$$

where，

V_i is the market value of the ith bond in the portfolio. It is a random variable.

V_P is the market value of the portfolio with a specific set of weights.

CV_m is the credit value at risk based on the transition metrics issued by the mth independent rating firm（we choose 5% as our significance level. The range of m is 1=S&P，2=Moody，3=Fitch）

τ is our defined parameter for describing the risk preference of different investors. In order to simulate the effective frontier, we use a function to repeatedly calculate different optimal portfolios at different level of risk preference $\tau_i = 0.002 * i^2$

μ is the mean

X_i is the weight of the ith asset in the portfolio and $\sum_{i=1}^{6} X_i = 100$.(Our initial investment is 10000USD)

4.2.4 The application of Simulated Annealing in our work

1 Solution space：

It is the group of all possible solution and it restrict the scope of choosing the initial solution and the new solution. Our constraint condition restrict the scope of solution must be within the range of 1 to 95 and should be integer with their sum of 100

$$S= \{(X_1, X_2, X_3, X_4, X_5, X_6) \mid X_1+X_2+X_3+X_4+X_5+X_6=100, X_i \in N\}$$

2 Objective function：

It is the mathematical description of the optimization problem. Usually it is constructed as the sum of several optimization targets.

$$\mathrm{Max}_X \{\mathrm{Min}_m (CV_m(V_P) + \tau * \mu_m(V_P))\} \qquad m = 1,2,3$$

We will find a set of X_i in solution space S to minimize the maximum C-VaR of the portfolio with these weights

3 Initial solution：

It is the starting point of the algorithm. Since that Simulated Annealing Algorithm is robust, ie the final solution is independent of the choice of the initial solution, we generate six random integer $X_i^{'}$ by Matlab，and use the following rules to standardize them to make sure their sum is 100.

$$\begin{cases} X_i = \left[\dfrac{100 * X_i^{'}}{\sum_{i=1}^{6} X_i^{'}} \right] & \text{for } X_1 \sim X_5 \\ X_6 = 100 - X_5 - X_4 - X_3 - X_2 - X_1 \end{cases}$$

Here, our initial solution is（8，8，3，24，45，12）

4 New solution’s generation and acceptance

Step 1, usually for the sake of convenience and time of calculation, new solution generation ways will be making small and simple modification of the existing solution like swap and so on. We choose the mechanism as follows:

Every time when the new solution is generated, we keep the first element of the solution and randomly generate the rest and randomly sort them together. Detailed algorithm is as follows:

```
Temp_x(1,1)=X(1,1);
i=2;
sum= Temp_x (1,1);
while(i<6)
    Temp_x (1,i)=randint(1,1,[1,94-sum+i]);
    sum = sum + Temp_x (1,i);
    i=i+1;
```

```
end
Temp_x (1,6)=100-sum;

j=randperm(6);
m=1;
while(m<7)
    X(1,m)= Temp_x (1,j(1,m));
    m=m+1;
end
```

Step 2, recalculate the corresponding objective function value at the new solution

$$f(P') = Max_X \{Min_m (CV_m (V_{P'}) + \tau * \mu_m (V_{P'}))\}$$

Step 3, determine whether or not to accept the new solution by an acceptance principal.

Here we use the most popular Metropo1is principal: If $f(P') \succ f(P)$ then accept P' as the new solution，otherwise accept it when $\exp\left(\frac{f(P')-f(P)}{t}\right) \prec random[0,1)$. Detailed Algorithm is as follows:

```
if(adapt_everbest<adapt_cur)
    adapt_everbest=adapt_cur
    solution_everbest=X
    CVaR_everbest=CVaR_cur
    mu_everbest=mu_cur
end
if(adapt_cur>=adapt_last)
    solution=X;
else
    if(rand>exp(adapt_last-adapt_cur)/T_cur)
        solution = X;
    end
end
```

Step 4, when new solution has been accepted, it should replace the current solution and modify the objective function value at the same time. Thus, we have made an itinerary process and do another round of experiment based on this. When the new solution is rejected by the acceptance principal, we continue next round of experiment based on the old value until the end of the annealing when temperature is declined to zero.

4.2.5 Output of the program

We only consider 50 different risk preferences because of the limit of the processing power. We show these optimal portfolios in the plane of two dimension of expected value and C-VaR with red dots.

- **Environment of the processing：**

Hardware：

IBM Corporation Intel (R) , Pentium (R), Processor 1500MHz, 1.50GHz, 内存 760MB

Software：

Microsoft Windows XP Home Edition 2002 版本，Service Pack 2

MATLAB Version 7.0.1.24704 (R14) Service Pack 1 September 13, 2004

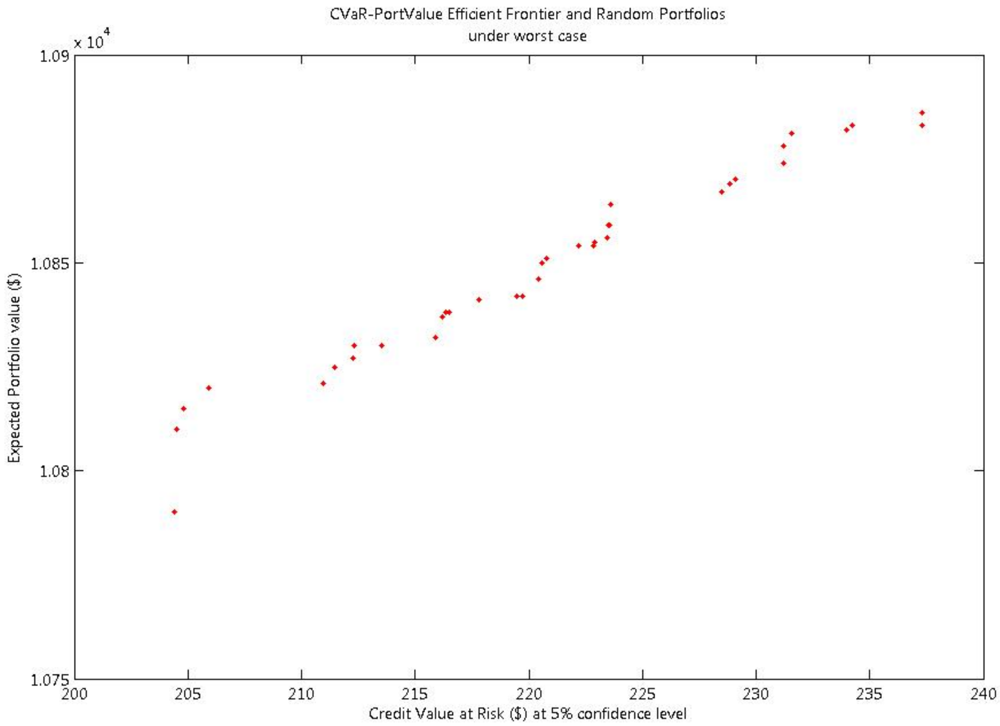
- **Processing time：**

5 hours12 minutes 33 seconds

- **Main program：**

```
 t=0.002;
point=1;
while(point<51)
        adapt_everbest=-inf;
        adapt_last=-inf;
        adapt_cur=-inf;
        T_ini=50;
        T_end=1;
        T_cur=T_ini;
        Markov_len=10;
        while(T_cur>T_end)
                n=1;
                while (n<Markov_len+1)
                        create;
                        adapting;
                        keepbest;
                        accept;
                        n=n+1;
                        fprintf('n= %f\n',n);
                end
                annealing;
        end
        CVaR(point,1)=CVaR_everbest;
        Mu(point,1)=mu_everbest;
        point=point+1
        t=0.002*point*point;
end
```

- **Output of the program：**



Graph 4.6 Optimal credit portfolios by different risk preferences under pessimistic decision

5 Summary and further discussion

In this paper, we will choose JP Morgan's credit metrics model to evaluate the portfolio's credit value-at-risk for the elaboration of our thesis and try to solve the problem of the confliction and negative impact on portfolio decisions brought by multiple rating metrics from different sources. We propose a method that borrows the idea of pessimistic decision to largely reduce the uncertainty by minimizing the maximum C-VaR using powerful algorithm simulated annealing for the optimization of credit portfolio. Under this method, we can at least secure our position and set us free from worrying about which source of rating is more creditable.

Key steps of the optimization process as follows:

1. Calculate the C-VaR for a portfolio with specific set of weights under the credit transition metrics issued by all independent rating firms.
2. Find the maximum C-VaR as the input for optimization process, using simulated annealing to minimize this C-VaR, getting the optimal portfolio.
3. For different risk preferences, we modify parameter τ and get a series of optimal portfolio allocation points consisting of the effective frontier like graph 4.6.

Of course, we still have several issues haven't touched in our work and new problems generating in our method. For instance, the period of measuring credit risk is one year (because we use annual asset return). However, in reality, some credit instruments have much short periods. Also, we haven't talk much about the marginal risk which is more concerned by the portfolio manager. While we leverage the powerful simulated annealing algorithm, we are subjected to the limited new solution generation mechanism instead of try more options and the length of Markov chains is set at the experience level, all of which might have some negative impact on our output.

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