A Cost of Capital Approach to Extrapolating an Implied Volatility Surface

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Abstract[1]

This paper develops an option pricing model which takes cost of capital concepts as its foundation rather than dynamic replication. The resulting model, called the '*C*' measure in this paper, is related to the family of Affine Jump Diffusion models that are well known in the finance literature so it is fairly easy to understand and implement. We argue that this is a reasonable model to use for estimating equity implied volatilities that are beyond the 5-10 year horizon that can typically be observed in today's capital markets. The paper concludes with a short discussion of how to grade from observable market data to the '*C*' measure.

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[1] The views and opinions expressed in this paper are those of the author and not the author's employer AEGON NV.

Introduction and Summary of Results

A problem faced by many life insurers today is that of putting a market consistent value on liability instruments that are longer than any comparable assets for which market data is available. This raises the issue of extrapolating yield curves, equity implied volatilities, correlations and other data beyond the available information. This paper focuses on the implied volatility problem but many of the ideas could be applied to the other problems as well.

One approach that immediately comes to mind is to take an industry standard model, a Heston model or a Heston model with jumps for example, and use the available market data to calibrate the model's parameters. Having calibrated the first 10 or so years of the model we simply assume the model applies in the later years and use it. We will call this approach "simple extrapolation" in this paper.

This extrapolation method has the advantages of being simple to understand and easy to implement. A potential disadvantage is that the parameters that produce a reasonable fit to near term market data may produce an unreasonable extrapolation.

The cost of capital method described here is a practical alternative to simple extrapolation. It is based on an approach endorsed by the CRO forum for valuing non-hedgeable risk. The basic idea is that if a risk cannot be hedged in the capital markets it should be valued using best estimate assumptions together with sufficient risk margins to pay for the cost of holding an appropriate amount of economic capital for the risk.

This new approach essentially derives a long term implied volatility assumption from first principles. In most situations it will do a better job of fitting observed volatility surfaces than the Black Scholes model but, only by chance, will it produce a good fit to observed market data. Additional work is then needed to engineer a model that fits both the current market and then grades into the longer term implied volatilities described here.

This paper will use a three step process to developing a cost of capital model which we illustrate here using both a "normal" insurance risk, mortality, and option pricing risk.

1) Develop a best estimate model. In the case of mortality risk this could be a deterministic or stochastic mortality table. For the option pricing problem we will take best estimate to mean the P, or real world, measure. Let σ be our best estimate of long term volatility and let μ be our best estimate of long term drift. These might come from a simple lognormal model or something more sophisticated.

2) Acknowledge the fact that even if the best estimate model is correct we can still have bad experience in any given year. The occurrence of a serious flu pandemic would be an example in the mortality case. Economic capital and margins need to be established to cover this risk. We will refer to this as contagion risk.

   For the option pricing problem we will take the analog of a contagion event to be an instantaneous jump in the equity markets $S \to JS$. Here J would be 60% if we were holding capital cover a 40% drop in the equity markets. One of this paper's key results is that the impact of adding contagion risk to the option pricing model is to raise long term implied volatilities from the best estimate σ to $\sqrt{\sigma^2 + 2(\mu - r)(J - 1 - \ln J)/(1 - J)}$. Here r a long term interest rate so $\mu - r$ is the long run equity premium.

3) As time evolves new information can arrive which causes us to revise one or more of the parameters in our best estimate model. We must hold economic capital and margins to cover a plausible shock to the best estimate model. Using mortality as an example this could be a shock to the mortality level or improvement trend. This will be called parameter risk in this paper.

   If our best estimate equity volatility were σ then this might entail holding capital to cover a jump in this parameter to something like $\sqrt{\sigma^2 + \Delta\sigma^2}$ or higher. Adding parameter risk to the model increases long term implied volatilities over and above the result obtained from contagion risk although we argue that contagion risk is the more material issue.

The main goal of this paper is to show how this three step approach can be applied to develop an option pricing model. Once the model is developed, and its properties understood, we argue that it is a reasonable approach to valuing options that can't be hedged directly in the capital markets. The final step of the paper is to consider the practical problem of grading from a market calibrated model to a cost of capital model.

Before going into the option pricing issues in more detail, it is appropriate to point out that all cost of capital models are vulnerable to the criticism that it is not always clear how a given parameter or assumption should be shocked . For simple life insurance or annuity products it is clear that mortality rates should be shocked up or down as the case may be. However, it is possible to engineer a product with a mix of mortality/longevity issues such that the nature of the risk varies by contract duration or possibly even market conditions. In this more general situation rigorous application of the cost of capital principles can lead to problems that require stochastic control concepts for their solution.

For the option pricing problem the analog of the mortality/longevity conundrum described above is whether we have a long/short equity exposure or whether the volatility exposure is convex/concave. Since most life insurers are long equity exposure and have convex liabilities there is a wide range of practical applications where this fundamental conundrum is not an issue. Problems requiring stochastic control concepts are therefore outside the scope of this paper.

Step 1 – The Best Estimate Model

We take the *P*, or real world, measure to be the analog of a best estimate mortality table. This could be a very simple model such as the standard log normal stock process $dS = \mu S dt + \sigma S dz$ or something more sophisticated such as an affine jump diffusion model. For simplicity of exposition we will use the the standard log normal model as a starting point in the examples that follow.

A high level formula for a best estimate value could be written as

$$V_0(t,S) = E_P[PV\ \text{"}Cash\ Flows\text{"}]$$

where present values are calculated using an appropriate risk free rate and *Cash Flows* are the projected cash flows of the instrument being valued.

For a vanilla put or call option a more mathematical formulation of the above idea is that the value $V_0(t,S)$ satisfies the partial differential equation

$$\frac{\partial V_0}{\partial t} + \mu S \frac{\partial V_0}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_0}{\partial S^2} = rV_0\,. \tag{1}$$

Here r is the risk free rate and appropriate boundary conditions at maturity of the option must be specified.

Step 2 – Contagion Risk

The analog of a contagion event for the option pricing problem is a finite jump $S \to JS$ where the jump factor J would be 60% if wanted to hold capital for a 40% drop in the equity markets.

The Responsible Speculator

Let's start by taking the point of view of a speculator who doesn't delta hedge. A responsible speculator should hold enough economic capital to cover the loss that would occur if such a market jump occurred. A high level formula that captures this idea is

$$\begin{aligned} V\ (t,S) &= E_P[PV\ \text{"}Cash\ Flows\text{"}+\text{"}Cost\ of\ \ Capital\text{"}] \\ &= E_P[PV\ \text{"}Cash\ Flows\text{"}+\pi\{V(t,JS)-V(t,S\}] \end{aligned}$$

This value differs from the best estimate in that it adds in the cost of holding capital for the jump risk. Here π is the cost of capital or risk premium that in investor expects to receive for putting up the risk capital. We can think of this value as a sum of the best estimate value $V_0(t,S)$ plus risk margins.

The formula is intended to capture the idea that cost of holding capital is being captured at all future points in time and market conditions. If the assumptions underlying the *P* measure model come true then an investor putting up the risk capital would earn, on average, the risk free rate plus the cost of capital π.

The model also assumes that gains and losses are continuously being trued up as time evolves. Gains are immediately paid out to the investor and losses are immediately replaced. The investor is willing to replace losses, or put up additional required capital, because there is always sufficient margin left on the balance sheet to guarantee a reasonable future return on the newly invested capital.

As written, the above formula is not very practical because it defines value in a circular fashion. It turns out that this valuation problem has a very practical solution. We will define a new risk adjusted process, called the *C* measure here, which is the *P* measure process augmented by a jump process $S \to JS$ where the instantaneous probability of a jump occurring in a small time interval *dt* is equal to the cost of capital rate π multiplied by the time interval.

A more mathematical way of seeing how the *C* measure concept comes about is to write down the basic valuation equation analogous to equation (1) above

$$\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - \pi[V(t,JS) - V(t,S)], \tag{2}$$

and then simply move all capital terms to left hand side to get

$$\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \pi[V(t,JS) - V(t,S)] = rV\ . \tag{3}$$

The way to interpret equation (2) is to say that, in the real world, we expect the quantity *V* to grow at the risk free rate while releasing sufficient margin to pay for the cost of capital. The mathematically equivalent formula (3) says that the quantity *V* has an expected rate of change equal to the risk free interest rate in the *C* measure world where the dynamics of the stock price are given by

$$dS = \mu S dt + \sigma S dz + (J-1)dN, \quad \Pr[dN = 1] = \pi dt.$$

In terms of the *C* measure, the value *V* defined above can then calculated as the expected present value

$$V\ (t,S) = E_C[PV\ "Cash\ Flows"\ ].$$

A minimal requirement for such a model to be market consistent is that it prices the stock process *S* back to itself. It is not hard to show[2] that this requirement forces us to set the cost of capital to equal the leveraged equity premium $\pi = (\mu - r)/(1-J)$. This return makes sense from the perspective of an investor putting up the risk capital since they have a leveraged exposure to the equity risk. The investor puts up a fraction $(1-J)$ of the equity position but takes 100% of the risk associated with that position.

As a simple example, assume the equity premium is $(\mu - r) = 4\%$ and $J = 60\%$ then the cost of capital for this risk would be $\pi = (\mu - r)/(1-J) = .04/(1-.6) = 10\%$.

The *C* measure model described above is actually a special case of Merton's (1973) Jump Diffusion model. There are many well documented[3] technical tools available for working with this model. In particular, for a vanilla put or call option that expires at time *T* the relationship between the value $V(t,S)$ described here and the best estimate value $V_0(t,S)$ is given by

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[2] Just demand the function $V(t,S)=S$ satisfy the equation (2).

[3] See, for example, Haug, E.G. "*The Complete Guide to Option Pricing Formulas*" , McGraw-Hill 1997.

$$V(t,S) = \exp[-\pi(T-t)]\sum_{n=0}^{\infty}\frac{[\pi(T-t)]^n}{n!}V_0(t,J^nS)\,. \tag{4}$$

This formula is easy to program so it is not hard to generate useful examples once a *P* measure model is chosen. We'll get more insight into what this means in the next section.

A reasonable criticism of the model described above is that it only appears to make sense for an instrument where $V(t,JS)-V(t,S)>0$ i.e. a put option or a long cash position. The next section will show why this model actually makes sense for any instrument which is convex.

The Responsible Hedger

We now take the point of view of someone who chooses to delta hedge an obligation. A delta hedger can hold less economic capital because a portion of the risk is hedged. However, the act of hedging puts the company in the *Q* measure effectively changing the expected return from μ to the risk free rate *r*. A high level formula for this new situation would be

$$V\ (t,S) = E_Q[PV\ "Cash\ Flows" + \hat{\pi}\{V(t,JS)-V(t,S)-(J-1)S\partial V/\partial S\}]\,.$$

The valuation equation is now given by

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2} = rV - \hat{\pi}[V(t,JS)-V(t,S)-(J-1)S\frac{\partial V}{\partial S}] \tag{5}$$

The responsible delta hedger using this model is saying that their *Q* measure calculation is basically right but they still hold enough capital to cover the un-hedged loss that would occur if a single large movement actually occurred. Adding this term addresses the very common criticism of the Black Scholes model that it ignores the possibility of large price movements. We also note that the adjustment makes the model more conservative as long as the instrument is convex.

We have used the symbol $\hat{\pi}$ for the cost of capital here because this risk is technically different from the one faced by the responsible speculator. In particular, this model prices the stock process *S* back to itself no matter what we assume for $\hat{\pi}$.

As before, there is a new risk adjusted measure which can be used to solve the valuation problem posed above. The solution is to start with the *Q* measure $dS = rSdt + \sigma Sdz$ and make two adjustments

1. Change the drift from the risk free rate *r* to $r + \hat{\pi}(1-J)$.
2. Add jumps $S \to JS$ with intensity $\hat{\pi}$ as before.

The risk adjusted stock process is now

$$dS = (r + \hat{\pi}(1-J))Sdt + \sigma Sdz + (J-1)dN, \quad \Pr[dN=1] = \hat{\pi}dt.$$

Again this is easily justified by rewriting equation (5) above with all capital terms on the left.

We now note that this model will agree with the C measure derived for the responsible speculator provided we assume the same cost of capital $\hat{\pi} = \pi = (\mu - r)/(1 - J)$. We conclude that speculators and hedgers can agree on value if they each hold capital for their respective unhedged risk and they agree that the cost of capital is $\pi = (\mu - r)/(1 - J)$. In the author's opinion this is a compelling argument for the use of the C measure as defined here.

The chart below shows three simulated price processes. The first is a standard log normal P measure scenario with $\mu = .08, \sigma = .15$. The second series is the corresponding the Black-Scholes Q measure scenario assuming the risk free rate is $r = .04$. The third series is an example of a C measure scenario which mimics P measure changes at all points in time except when a jump occurs. In this particular scenario jumps occur near years 8 and 21 with the result that the Q and C measures end up at very similar points after 25 years.

With the parameter choices made above the cost of capital is $\pi = (.08 - .04)/(1 - .6) = 10\%$ and the expected number of jumps over a 25 year period is then 2.5. Observing exactly two jumps is therefore a relatively high probability outcome.

We can get a high level sense of how this model differs from the standard Black-Scholes model by starting with equation (5) above and then approximating the cost of capital by a Taylor series

$$\pi[V(t, JS) - V(t, S) - (J - 1)S\frac{\partial V}{\partial S}] \approx \frac{1}{2}\pi(J - 1)^2 S^2 \frac{\partial^2 V}{\partial S^2}.$$

The valuation equation can then be rewritten as

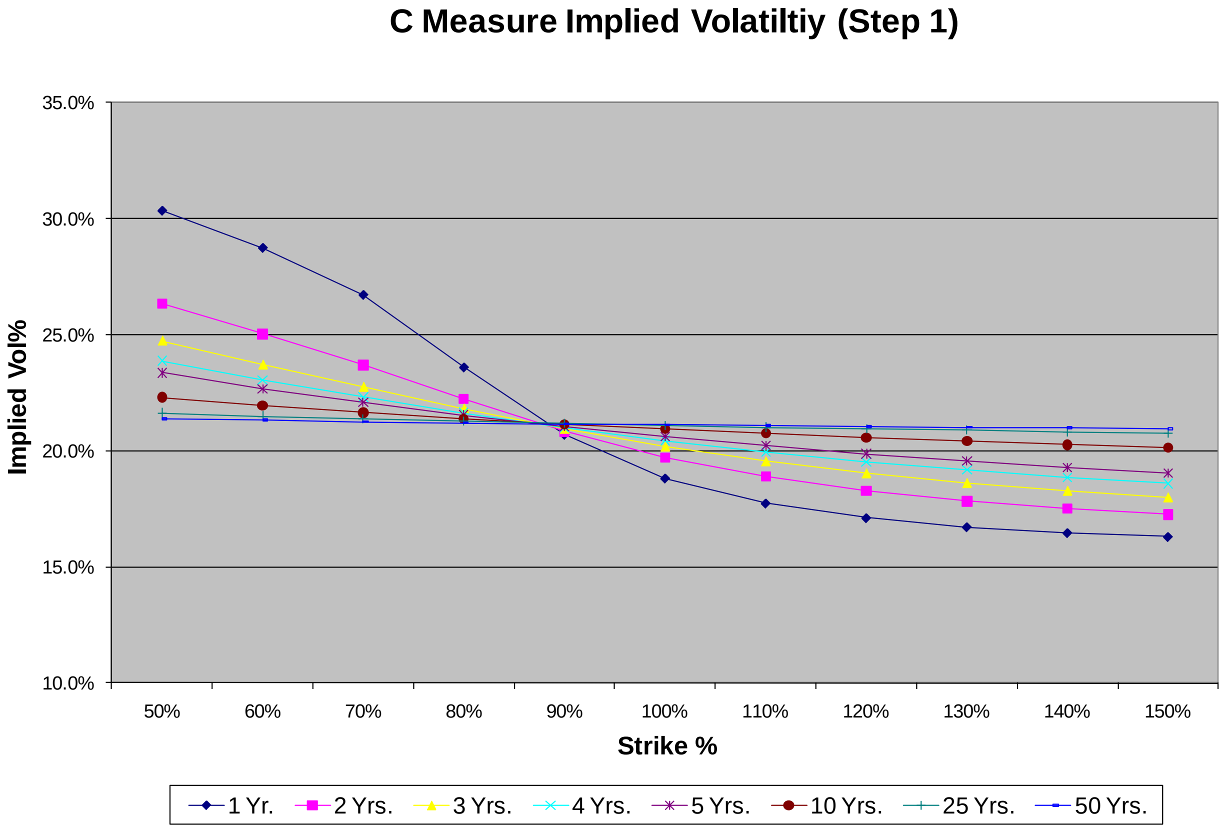
$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}[\sigma^2 + \pi(J-1)^2]S^2\frac{\partial^2 V}{\partial S^2} - rV \approx 0.$$

Adding this cost of capital to the Black Scholes equation is therefore roughly equivalent to using the Black Scholes model with an implied volatility give by the simple formula

$$\begin{aligned}\sigma_{imp}^2 &\approx \sigma^2 + \pi(1-J)^2, \\ &= \sigma^2 + (\mu - r)(1-J).\end{aligned}$$

In practice, this approximation is not very good at short durations and it slightly underestimates implied volatility at longer durations. A more detailed argument outlined in the appendix shows that $\sigma_{imp}^2 \approx \sigma^2 + 2\pi(J - 1 - \ln J)$ is a better formula approximation.

The graph below shows the entire volatility surface for the *C* measure model using the yield curve at Dec. 31, 2008, a dividend rate of 2%, a jump factor of 60% and a cost of capital equal to 10%. For *P* measure the graph assumes the standard lognormal model with $\mu = .08, \sigma = .15$.



For each maturity and strike put option values were calculated using the series expansion (4). We then solved for the volatility assumption that would produce the same value using the Black-Scholes model.

Three points worth noting at this stage of the model's development are

- The model exhibits the phenomenon of "skew" at shorter maturities where implied volatilities are a decreasing function of the strike price when the option is close to being at the money.

- The surface is almost flat by the time we are out 50 years. The implied volatility varies from 21.4% to 21.0% as the strike price ranges from 50% to 150%. This is consistent with the formula approximation

  $$\sigma_{imp}^2 \approx (.15)^2 + .10 \times 2(J - 1 - \ln J) \approx (.21)^2$$

  described earlier. The model has added a risk premium of about 600 vol points.

- The model does not exhibit the kind of "smile" that is observed in actual volatility surfaces. One reason for this is the use of the lognormal model as our best estimate. Had we started with a stochastic volatility model, the Heston model for example, the resulting *C* measure model would exhibit more smile at shorter durations[4].

Step 3 – Parameter Risk

The model summarized briefly above has two key parameters which are the best estimate volatility σ and the long run equity premium $\mu - r$. If there were no uncertainty in the best estimate model, or its parameters, then $\sigma_{imp}^2 \approx \sigma^2 + 2(\mu - r)(J - 1 - \ln J)/(1 - J)$ might be an appropriate long run implied volatility assumption as determined by the cost of capital method.

A wide range of models have been proposed to explain real world volatility. In the finance literature the Heston model is well known while regime switching models[5] are common in the actuarial literature. No matter what model we pick we have to estimate parameters and, as time evolves, new information can arrive which causes us to change our parameter estimates.

The issue is illustrated by the chart below which shows realized volatility by calendar year for the S&P 500 for the 60 year period beginning in 1950.

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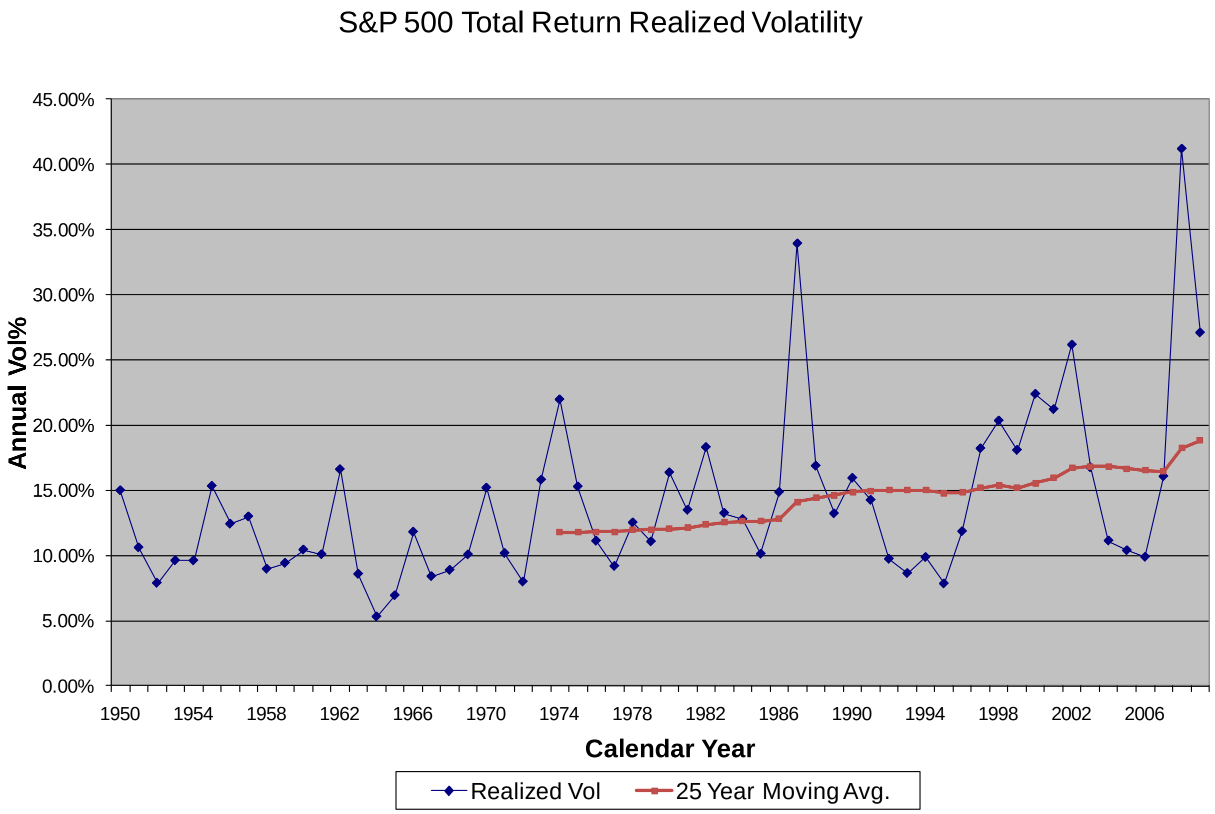
[4] For more background on the concepts of skew and smile see Gatheral, J. "*The Volatility Surface*: *A Practitioner's Guide*", Wiley (2006).

[5] Hardy, M.R., " *A Regime Switching Model of Long Term Stock Returns*", North American Actuarial Journal (2001).



The chart also shows the 25 year trailing average starting at the end of 1974 which is a very simple method for developing a “best estimate” long term volatility assumption.

While there is no unique methodology for turning historical data into a best estimate assumption it is clear that any reasonable combination of *P* measure model and parameter estimation method would have resulted in revisions to the long term parameter assumptions as time evolved.

One way to deal with this is to hold sufficient economic capital that we can cover the loss that would occur if the liability were revalued using a revised set of parameter values. We’ll illustrate this idea in the case where the *P* measure model is just the standard lognormal. We have a best estimate long term volatility assumption of σ^2 and we want to hold capital and margins to cover a shocked assumption $\hat{\sigma}^2 = \sigma^2 + \Delta\sigma^2$. Here $\Delta\sigma^2$ is a plausible (99.5%) shock to the volatility assumption that could result from new information arriving in the course of the next year.

As an example, suppose we are at the end of 2007 and our best estimate of $\sigma = .15$ is based on the prior 25 years of history. As the financial crisis hit in 2008 realized volatility jumped to about 41%. A revised best estimate might then have been something like

$$\begin{aligned}\hat{\sigma}^2 &= \frac{24(.15)^2 + (.41)^2}{25} \approx (.17)^2 \\ &\approx (.15)^2 + (.08)^2\end{aligned}$$

This kind of analysis suggests that a one year shock in the range of $(.05)^2 < \Delta\sigma^2 < (.10)^2$ would be reasonable.

Taking the responsible speculator’s point of view, a high level formula that captures this idea would be to write

$$V\ (t,S) = E_P[PV\ \text{"}Cash\ Flows\text{"} + \pi\{V(t,JS) - V(t,S\} + \tilde{\pi}\{\hat{V}(t,S) - V(t,S)\}]\,.$$

Here $\hat{V}(t,S)$ is a value calculated using shocked volatility and $\tilde{\pi}$ is a new cost of capital appropriate for this new risk. If $\hat{P}$ is the shocked P measure then we would compute the shocked value $\hat{V}(t,S)$ using

$$\hat{V}\ (t,S) = E_{\hat{P}}[PV\ \text{"}Cash\ Flows\text{"} + \pi\{\hat{V}(t,JS) - \hat{V}(t,S\} + \tilde{\pi}\{\hat{\hat{V}}(t,S) - \hat{V}(t,S)\}]\,.$$

This formula assumes we take the same approach to contagion risk when calculating $\hat{V}(t,S)$ that we did in the base case. The formula also indicates that we would continue to hold capital $\{\hat{\hat{V}}(t,S) - \hat{V}(t,S)\}$ for parameter risk in the shocked world. The only reason to omit such a term from the calculation of $\hat{V}(t,S)$ is because we thought the shocked parameter value $\hat{\sigma}^2 = \sigma^2 + \Delta\sigma^2$ could never get any worse. If this is not the case then double, triple, …,*n* times shocked worlds need to be considered, in theory.

If this seems like over engineering, we would agree and we will shortly show how the technicalities can be simplified significantly. However, it is worth emphasizing why such a model structure makes sense. If the shocked value $\hat{V}(t,S)$ contains no margins then we are not in a position to attract new capital if we found ourselves in the shocked world and the previous capital $\{\hat{V}(t,S) - V(t,S)\}$ was used up by an assumption change. In order to attract a capital infusion of $\{\hat{\hat{V}}(t,S) - \hat{V}(t,S)\}$ the shocked value $\hat{V}(t,S)$ must contain sufficient margin to compensate an investor for taking the risk.

The technical solution to the valuation problem we have just defined is a regime switching model where volatility starts out equal to the best estimate value and then randomly jumps from one level to the next with a transition rate equal to the cost of holding capital for parameter risk.

In theory, this requires us to specify an infinite hierarchy of volatility levels $\sigma^2 \to \sigma^2 + \Delta\sigma^2 \to \ldots \to \hat{\sigma}^2_\infty$ along with a cost of capital $\tilde{\pi}$. The *C* measure introduced earlier is now augmented by allowing the volatility parameter to jump randomly up the shock hierarchy with transition intensity $\tilde{\pi}$.

To calculate the shocked value $\hat{V}(t,S)$ we need a shocked *C* measure $\hat{C}$ which is the same as the *C* measure except the volatility assumption starts out in the first shocked level $\hat{\sigma}^2 = \sigma^2 + \Delta\sigma^2$ and then jumps randomly up the hierarchy from there.

In theory, the discussion above completes the description of the *C* measure. Had we started with a more sophisticated *P* measure model the same basic ideas could have been developed except that the specific parameters subject to regime switching might be different from what makes sense in the standard lognormal model.

As a practical matter, we need a way to justify a set of assumptions and, if possible, simplify the actual calculations. Most practitioners would consider implementing the regime switching model, as specified, to be over engineering.

To simplify the regime switching model we make an additional assumption.

> The shock hierarchy has a simple geometric structure governed by a parameter $0 < \alpha < 1$. In terms of this parameter the *n*'th level in the volatility hierarchy is assumed to be
>
> $$\hat{\sigma}_n{}^2 = \sigma^2 + \Delta\sigma^2 \frac{1-\alpha^n}{1-\alpha}.$$

The parameter α is then chosen so that the ultimate volatility level $\sigma_\infty^2 = \sigma^2 + \Delta\sigma^2/(1-\alpha)$ makes sense. The historical evidence shown earlier shows that it is very difficult to get S&P 500 realized volatility over a 10 year, or longer, period to exceed 20%. This supports an alpha factor of about 50%.

If the cost of capital is $\tilde{\pi}$ then the *C* measure expected squared forward volatility, *T* years from the valuation date, is[6]

$$\overline{\sigma}^{\,2}(T) = \sigma^2 + \frac{\Delta\sigma^2}{1-\alpha}[1-\exp(-\tilde{\pi}T(1-\alpha))]$$

which corresponds to the spot volatility

$$s^2(T) = \frac{1}{T}\int_0^T \overline{\sigma}^{\,2}(v)dv = \sigma^2 + \frac{\Delta\sigma^2}{1-\alpha}[1-\frac{1-\exp(-\tilde{\pi}T(1-\alpha))}{\tilde{\pi}T(1-\alpha)}].$$

This is a good approximation to the regime switching model's implied volatility for all but very short maturities.

If we now put the contagion and parameter risk pieces together we find that, for long dated options, the cost of capital model suggests the following approximate formula for implied volatility

$$\begin{aligned} \sigma_{imp}^2 &\approx \sigma^2 + 2\pi(J-1-\ln J) + \frac{\Delta\sigma^2}{1-\alpha}, \\ &= (.15)^2 + 2\times.10\times(.6-1-\ln .6) + (.08)^2/(1-.5), \\ &= (.240)^2 \end{aligned}$$

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[6] This follows from the fact that number of regime changes $N(T)$ has a Poisson distribution with mean $\tilde{\pi}T$ so that $E[\alpha^{N(T)}] = \exp[-\tilde{\pi}(1-\alpha)T]$. This is one reason for choosing the geometric hierarchy.

The end result of the process would appear to add roughly 900 vol points to the best estimate volatility of 15%. As the table below shows it takes more than 50 years for this long term implied volatility to be reached if we assume a reasonable cost of capital for parameter risk.

Each line in the table below is designed to illustrate the impact of a specific set of *C* measure parameters on at the money implied volatilities for maturities of 10, 25 and 50 years.

	Contagion Shock			**Parameter Shock**			**At the Money Implied Vol %**		
	σ	J	π	$\Delta\sigma$	α	$\pi\sim$	**10**	**25**	**50**
1	15.0%	60.0%	10.0%	8.0%	50.0%	4.0%	21.1%	21.6%	22.1%
2	15.0%	60.0%	10.0%	0.0%	50.0%	4.0%	20.8%	20.9%	21.0%
3	18.0%	60.0%	10.0%	8.0%	50.0%	4.0%	23.3%	23.7%	24.2%
4	15.0%	50.0%	8.0%	8.0%	50.0%	4.0%	22.8%	23.3%	23.8%
5	15.0%	60.0%	15.0%	8.0%	50.0%	4.0%	23.4%	23.9%	24.3%
6	15.0%	60.0%	10.0%	10.0%	50.0%	4.0%	21.3%	21.9%	22.6%
7	15.0%	60.0%	10.0%	8.0%	75.0%	4.0%	21.1%	21.6%	22.2%
8	15.0%	60.0%	10.0%	8.0%	50.0%	10.0%	21.4%	22.2%	22.8%

The first line of the table shows what happens if we assume that an equity premium of 4% is appropriate for parameter risk. This rate seems reasonable given that parameter risk is not leveraged like contagion risk and so it is more like an insurance risk.

With a 4% probability of stepping up the shock hierarchy each year we have experienced, on average, only two regime changes in 50 years. This would lead to an expected forward volatility of about 18% and a spot volatility of 16.5% under the regime switching model.

The last line in the table above shows what happens if we increase the cost of parameter risk capital from 4% to 10%. The effect is not huge. In fact, an important high level conclusion is that contagion risk issues are more important than parameter risk issues.

Line 2 in the table simply turns off the parameter shock. This shows that the impact on 50 year implied volatility of adding parameter risk is more like 100 vol points rather than 300 points as indicated earlier.

Line 3 shows what happens if we change the underlying best estimate volatility from 15% to 18%. Fifty year implied volatilities go up by about 210 vol points. There is a diversification effect at work here which is why the implied volatility did not go up by 300 vol points.

Line 4 shows what happens if the contagion capital shock is changed from 40% to 50% while we leave the equity risk premium fixed at 4%. The leveraged cost of capital is now 8%. This does increase implied volatilities in a material way though not as much as in Line 3.

Line 5 shows what happens if we change the cost of contagion risk capital while leaving the capital shock at 40%. This is equivalent to assuming that the equity risk premium is raised from 4% to 9%. This produces results very similar to Line 3.

Lines 6,7, and 8 test the model's sensitivity to changes in the parameters controlling the cost of parameter risk. Not surprisingly, the first level shock in Line 6 is the most significant issue.

Step 4 – The Case for the *C* Measure

The first argument to support the use of the *C* measure is that it represents a cost of manufacturing an option that depends only on a small number of fundamental economic assumptions. All of these assumptions are fairly transparent and easily subjected to scrutiny. Furthermore, this has been done within the context of high level principles already accepted by the CRO Forum as reasonable for valuing non-hedgeable risk.

The model does not assume delta hedging but it is not inconsistent with delta hedging either. In fact the model addresses one of the principle criticisms of the Black- Scholes model in that it recognizes that delta hedging, if implemented, is an imperfect process.

The technicalities of the model are not onerous. Anyone familiar with basic Black-Scholes concepts and Merton's jump diffusion extension can come to grips with the technical details at whatever level is necessary.

It is not necessary to take the standard lognormal model as the starting point. This paper takes this approach to simplify the presentation of the relevant new ideas. Once can easily start with a more sophisticated *P* measure model and then adjust it for contagion and parameter risk as we have done here.

Step 5 - Grading from a Market Calibrated Model to the *C* Measure

The model described in this paper can be used in a number of different ways. The simplest way is to use the ideas developed here to justify the choice of parameters in some other model. How this works in practice will depend on the other model being used so we can't comment further here.

At the other end of the spectrum one can imagine trying to come up with a version of the *C* measure model that can be calibrated to market data. Some of the challenges that must be addressed can be seen in the short history of 10 year At The Money (ATM) options in the table below.

S&P 500 10 Year
ATM Implied Vol

6/30/2009	31.1%
12/31/2008	34.9%
12/31/2007	27.5%
12/31/2006	22.4%
12/30/2005	24.0%
12/31/2004	19.7%
12/31/2003	18.0%

These volatilities have not borne much resemblance to the *C* measure estimate of 22% since the end of 2006. More generally, the volatility surfaces that have been observed in the market during the last few years exhibit more skew and smile than can be explained by the simple *C* measure model discussed in this paper.

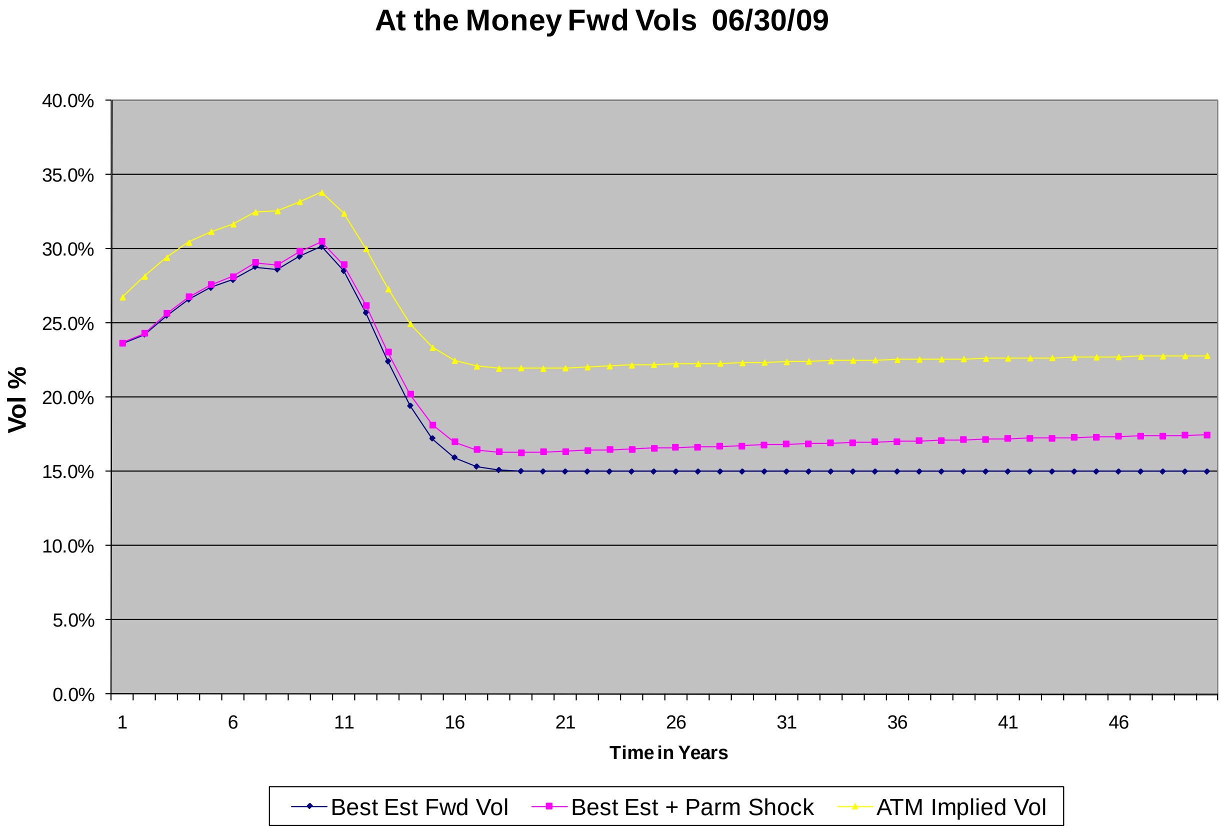
If a high degree of fit is required then we would need to add elements such as stochastic volatility and/or jumps to the *P* measure model before we started adjusting for the cost of capital. See Chapter 5 of Gatheral[7] for more details of how this might work.

If we have a more limited objective, matching At the Money implied volatilities only, for example then a simpler approach is possible. One idea is to allow the best estimate to be a deterministic function of time determined as follows

1. Choose the best estimate forward volatilty in years 1-10 so as that the *C* measure model's at the money implied volatilities match the market.

2. Grade the forward volatilities in years 11+ into the assumed true best estimate (e.g. 15%).

The chart below shows an example of this process as of June 30, 2009 where exponential grading at an annual rate of 15% was used.



There is no real science in the choice of grading scheme but most people would probably find grading over a 5-10 year period reasonable.

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[7] Gather al, J. "*The Volatility Surface A Practitioner's Guide*", Wiley (2006)

An alternative is to allow the forward rate assumption to jump discontinuously from the last market determined rate to *C* measure value. This may seem extreme but it is consistent with the idea of using the *C* measure to put a value, at time 10, on all cash flows beyond the market's horizon. Market discounting is then used to discount those values from time 10 to the valuation date.

While neither of the two approaches outlined above is "right" they do have different risk management implications. Allowing discontinuous forward rates will generally lead to values that are less volatile and more easily hedged with real market instruments.

Conclusion

This paper has developed an option pricing model from first principles using cost of capital concepts as a foundation. In the author's opinion the most useful aspect of the model is its ability to defend a long term implied volatility assumption. This long term volatility depends on a small number of fairly transparent macroeconomic inputs and is independent of many modeling details.

We have also shown that modeling parameter risk naturally leads to a regime switching model for the uncertain parameters and we have given a simple example.

APPENDIX

The main purpose of this appendix is to derive the formula approximation for long term implied volatility used in Step 2 of this paper

$$\sigma_{imp}^2 \approx \sigma^2 + 2\pi(J-1-\ln J) = \sigma^2 + (\mu - r)(J-1-\ln J)/(1-J).$$

The key idea is that, over longer periods of time, the law of large numbers allows us to approximate any reasonable *P* measure process by a lognormal model. Adding the simple jumps required by the *C* measure simply gives a different lognormal approximation.

Suppose that, under the *P* measure, we have the approximation

$$S(t) = S(0)\exp[(\mu - \sigma^2/2)t + \sigma Z(t)].$$

Going to the *C* measure means we now have

$$\begin{aligned} S(t) &= S(0)\exp[(\mu - \sigma^2/2)t + \sigma Z(t)]J^{N(t)}, \\ &= S(0)\exp[(\mu - \sigma^2/2)t + \sigma Z(t) + \ln(J)N(t)]. \end{aligned}$$

Here $N(t)$ is a Poisson process with mean $\pi t = (\mu - r)t/(1-J)$.

Now use $\mu = r + \pi(1-J)$ to write the stock process as

$$S(t) = S(0)\exp[(r - \sigma^2/2)t + \sigma Z(t)]\exp[\pi(1-J)t + \ln(J)N(t)].$$

This shows that the *C* measure process is, roughly, the product of a Black Scholes *Q* measure process with volatility σ and a second, independent, process $\exp[\pi(1-J)t + \ln(J)N(t)]$. This second process always has a mean of 1so we expect that, for large enough *t*, we should have an approximation of the form

$$\exp[\pi(1-J)t + \ln(J)N(t)] \approx \exp[-\tilde{\sigma}^2 t/2 + \tilde{\sigma}\tilde{Z}(t)], \tag{*}$$

where $\tilde{Z}(t)$ is a new Weiner process independent of $Z(t)$. For large *t* we know that $N(t) \to \pi t + \sqrt{\pi}\tilde{Z}(t)$ which suggests that $\tilde{\sigma} = \sqrt{\pi}\ln(J)$ might be a good answer. This approximation suggests that $\sigma_{imp}^2 \approx \sigma^2 + \pi\ln(J)^2$ ought to be a reasonable long term implied volatility. Practical testing shows that this formula tends to overstate the implied volatility for long dated vanilla options.

A better approximation is obtained by choosing $\tilde{\sigma}$ so that the log means of each term in equation (*) above are equal i.e. we choose $\tilde{\sigma}$ so that

$$\begin{aligned}
&E[\pi(1-J)t + \ln(J)N(t)] = E[-\tilde{\sigma}^2 t/2 + \tilde{\sigma}\tilde{Z}(t)], \\
&\Leftrightarrow \pi(1-J)t + \ln(J)\pi t = -\tilde{\sigma}^2 t/2, \\
&\Rightarrow \tilde{\sigma}^2 = 2\pi(J - 1 - \ln J).
\end{aligned}$$

In conclusion this paper has actually derived three formula approximations for long term implied volatility. These are

$$\sigma_{imp}^2 \approx \sigma^2 + \pi(1-J)^2 \leq \sigma^2 + 2\pi(J - 1 - \ln J) \leq \sigma^2 + \pi\ln(J)^2.$$

Empirical testing shows that the middle formula works best for implied volatility of vanilla options.

The other two expressions are reasonable approximations for the fair value of a variance swap contract under the *C* measure. Each formula applies to a slightly different definition of realized volatility.