Optimal Layers for Catastrophe Reinsurance

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Agenda

- Introduction
- Optimal reinsurance: academics
- Optimal reinsurance: RAROC
- Optimal reinsurance: our method
- A case study
- Conclusions
- Q&A
1. Introduction

Reinsurance decision is a balance between cost and benefit

- Cost: reinsurance premium – loss recovered
- Benefit: risk reduction
  - Stable income stream over time
  - Protection against extreme events
  - Reduce likelihood of being downgraded
1. Introduction

How to measure risk reduction

- Variance and standard deviation
- Not downside risk measures
- Desirable swings are also treated as risk

- VaR (Value-at-Risk), TVaR, XTVaR
  - VaR: predetermined percentile point. PML (probable maximum loss per event) is a VaR measure at event level
  - TVaR: expected value when loss > VAR
  - XTVaR: TVaR-mean
1. Introduction

How to measure risk reduction

- Lower partial moment and downside variance

\[ LPM(L \mid T, k) = \int_{T}^{\infty} (L - T)^k dF(L) \]

- \( T \) is the maximum acceptable losses, benchmark for “downside”
- \( k \) is the risk perception parameter to large losses, the higher the \( k \), the stronger risk aversion to large losses
- When \( k=1 \) and \( T \) is the 99th percentile of loss, LPM is equal to 0.01*VaR
- When \( K=2 \) and \( T \) is the mean, LPM is semi-variance
- When \( K=2 \) and \( T \) is the target, LPM is downside variance
2. Optimal reinsurance: academics

2. Optimal reinsurance: academics

- Cat reinsurance has zero correlation with market index, and therefore zero beta in CAPM.
- Because of zero beta, reinsurance premium should be a dollar-to-dollar trade of loss recovered.
- Reinsurance reduces risk at zero cost. Therefore optimizing profit-risk tradeoff implies minimizing risk
  - buy largest possible protection without budget constraints
  - buy highest possible retention with budget constraints
2. Optimal reinsurance: academics

Academic Assumption

Profit

Risk

U1

U2

U3

A

B
2. Optimal reinsurance: academics

Those studies do not help practitioners

- Reinsurance is costly.
  - Reinsurers need to hold a large amount of capital and require a market return on such a capital.
  - Reinsurance premium/Loss recovered can be over 10 in reality
- No reinsurers can fully diversify away cat risk
- Only consider the risk side of equation and ignore cost side.
3. Optimal reinsurance: RAROC

RAROC (Risk-adjusted return on capital) approach is popular in practice

- Economic capital (EC) covers extreme loss scenarios
- Reinsurance cost = reinsurance premium – expected recovery
- Capital Saving = EC w/o reinsurance – EC w reinsurance
- Cost of Risk Capital (CORC) = Reinsurance cost / Capital Saving
- CORC balances profit (numerator) and risk (denominator)
3. Optimal reinsurance: RAROC

- No universal definition of economic capital
- Use VaR or TVaR to measure risk
  - Only consider extreme scenarios.
  - Linear risk perception.
4. Optimal Reinsurance: DRAP Approach

Downside Risk-adjusted Profit (DRAP)

\[
DRAP = \text{Mean}(r) - \theta \times LPM(r \mid T, k)
\]

\[
LPM(r \mid T, k) = \int_{-\infty}^{T} (T - r)^k dF(r)
\]

- \(r\) is underwriting profit rate
- \(\theta\) is the risk aversion coefficient
- \(T\) is the benchmark for downside
- \(K\) measures the increasing risk perception towards large losses
4. Optimal Reinsurance: DRAP Approach

Loss Recovery

\[ G(x_i, R, L) = \begin{cases} 
0 & \text{if } x_i \leq R \\
(x_i - R) \phi & \text{if } R < x_i \leq R + L \\
L \phi & \text{if } x_i > R + L 
\end{cases} \]

- R is retention
- L is the limit
- \( \Phi \) is the coverage percentage
- \( x_i \) is cat loss from the ith event
4. Optimal Reinsurance: DRAP Approach

Underwriting profit

\[ r = 1 - \frac{\text{EXP} + Y + \text{RP}(R, L)}{\text{EP}} - \frac{\sum_{i=1}^{N} x_i - G(x_i, R, L) + \text{RI}(x_i, R, L)}{\text{EP}} \]

- EP: gross earned premium
- EXP: expense
- Y: non-cat losses
- RP(R, L): reinsurance premium
- RI(xi, R, L): reinstatement premium
- N: number of cat events
4. Optimal Reinsurance: DRAP Approach

\[ \max_{R,L} \text{Mean}(r) - \theta \ast \text{LPM}(r | T, k) \]

AB is efficient frontier
U1, U2, U3 are utility curves
C is the optimal reinsurance that maximizes DRAP
4. Optimal Reinsurance: DRAP Approach

Advantages to conventional mean-variance studies in academics

- An ERM approach.
  - Considers both catastrophe and non-catastrophe losses simultaneously
  - Overall profitability impacts layer selection. High profitability enhances an insurer’s ability to retain more cat risk.

- Use a downside risk measure (LPM) other than two-side risk measure (variance)
4. Optimal Reinsurance: DRAP Approach

Theta estimations

\[ DRAP = \text{Mean}(r) - \theta \times LPM(r \mid T, k) \]

- Theta may not be constant by the size of loss
- Theta is time variant
- Theta varies by individual institution
- How much management is willing to pay to mitigate risk?
- How much do investors require to take the risk?

- Index risk premium = index return – risk free rate
- Insurance risk premium = insurance return-risk free rate
- Cat risk premium = cat bond yield- expected loss-risk free rate
4. Optimal Reinsurance: DRAP Approach

K and T estimations

\[ LPM(r \mid T, k) = \int_{-\infty}^{T} (T - r)^k dF(r) \]

- \( k \) may not be constant by the size of loss
  - For smaller loss, loss perception is close to 1, \( k = 1 \); for severe loss, \( k > 1 \)
  - Academic tradition: \( k = 2 \)
- \( T \) is the benchmark for “downside”
  - Zero: underwriting loss is risk
  - Zero ROE: underwriting loss larger than investment income is risk
  - Large negative: severe loss is treated as risk
5. Case Study

A hypothetical company

- Gross earned premium from all lines: 10 billion
- Expense ratio: 33%
- Lognormal non-cat loss from actual data
  mean=5.91 billion; std=402 million
- Lognormal cat loss estimated from AIR data
  - mean # of event=39.7; std=4.45
  - mean loss from an event=10.02 million; std=50.77 million
  - total annual cat loss mean=398 million; std=323 million
5. Case Study

- $K=2$
- $T=0\%$
- Theta is tested at 16.71, 22.28, and 27.85, which represents that primary insurer would like to pay 30\%, 40\%, and 50\% of gross profit to hedge downside risk, respectively.
- UW profit without Insurance is 3.92\%
- Variance 0.263\%
- Downside variance is 0.07\% ($T=0\%$)
- Probability of underwriting loss is 18.41\%
- Probability of severe loss ($<-15\%$) is 0.48\%
## 5. Case Study

### Reinsurance quotes (million)

<table>
<thead>
<tr>
<th>Retention</th>
<th>Upper Bound of Layer</th>
<th>Reinsurance Limit</th>
<th>Reinsurance Price</th>
<th>Rate-on-line</th>
</tr>
</thead>
<tbody>
<tr>
<td>305</td>
<td>420</td>
<td>115</td>
<td>20.8</td>
<td>18.09%</td>
</tr>
<tr>
<td>420</td>
<td>610</td>
<td>190</td>
<td>21.7</td>
<td>11.42%</td>
</tr>
<tr>
<td>610</td>
<td>915</td>
<td>305</td>
<td>19.8</td>
<td>6.50%</td>
</tr>
<tr>
<td>610</td>
<td>1,030</td>
<td>420</td>
<td>25.2</td>
<td>5.99%</td>
</tr>
<tr>
<td>1,030</td>
<td>1,800</td>
<td>770</td>
<td>28.7</td>
<td>3.72%</td>
</tr>
<tr>
<td>1,800</td>
<td>3,050</td>
<td>1,250</td>
<td>39.1</td>
<td>3.13%</td>
</tr>
</tbody>
</table>
## 5. Case Study

### Recoveries and penetrations by layers

<table>
<thead>
<tr>
<th>Retention (million)</th>
<th>Upper Limit (million)</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Recovery/reinsurance Premium</th>
<th>Penetration Probability</th>
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<tbody>
<tr>
<td>305</td>
<td>420</td>
<td>8,859,074</td>
<td>29,491,239</td>
<td>42.59%</td>
<td>10.18%</td>
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<tr>
<td>420</td>
<td>610</td>
<td>8,045,968</td>
<td>35,917,439</td>
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<td>915</td>
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<td>610</td>
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<td>7,923,052</td>
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<tr>
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<td>1,800</td>
<td>4,858,545</td>
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<td>1.11%</td>
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<tr>
<td>1,800</td>
<td>3,050</td>
<td>2,573,573</td>
<td>48,827,021</td>
<td>6.58%</td>
<td>0.40%</td>
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</table>
5. Case Study

Reinsurance Price Curves Fitting

- \((x_1, x_2)\) represents reinsurance layer
- \(f(x)\) represents rate-on-line

\[
\begin{align*}
p(x_1, x_2) &= \int_{x_1}^{x_2} f(x) \, dx \\
&= \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 \log(x) + \beta_4 x^{-1}
\end{align*}
\]

- Add quadratic term, logarithm, and inverse term to reflect nonlinear relations

\[
egin{align*}
p(x_1, x_2) &= \beta_0 (x_2 - x_1) + \frac{1}{2} \beta_1 (x_2^2 - x_1^2) + \frac{1}{3} \beta_2 (x_2^3 - x_1^3) \\
&\quad + \beta_3 (x_2 \log(x_2) - x_1 \log(x_1)) + \beta_4 (\log(x_2) - \log(x_1))
\end{align*}
\]
## 5. Case Study

### Reinsurance Price Fitting

<table>
<thead>
<tr>
<th>Retention</th>
<th>Upper Bound of Layer</th>
<th>Reinsurance Limit</th>
<th>Reinsurance Price</th>
<th>Rate-on-line</th>
<th>Fitted rate</th>
<th>Fitted Rate-on-line</th>
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<tbody>
<tr>
<td>305</td>
<td>420</td>
<td>115</td>
<td>20.8</td>
<td>18.09%</td>
<td>20.84</td>
<td>18.12%</td>
</tr>
<tr>
<td>420</td>
<td>610</td>
<td>190</td>
<td>21.7</td>
<td>11.42%</td>
<td>21.69</td>
<td>11.41%</td>
</tr>
<tr>
<td>610</td>
<td>915</td>
<td>305</td>
<td>19.8</td>
<td>6.50%</td>
<td>19.87</td>
<td>6.51%</td>
</tr>
<tr>
<td>610</td>
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<td>420</td>
<td>25.2</td>
<td>5.99%</td>
<td>25.18</td>
<td>6.00%</td>
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<tr>
<td>1,030</td>
<td>1,800</td>
<td>770</td>
<td>28.7</td>
<td>3.72%</td>
<td>28.73</td>
<td>3.73%</td>
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<td>1,800</td>
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<td>1,250</td>
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<td>39.10</td>
<td>3.13%</td>
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<td>610</td>
<td>305</td>
<td>42.5</td>
<td>13.93%</td>
<td>42.52</td>
<td>13.94%</td>
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<td>915</td>
<td>610</td>
<td>62.3</td>
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<td>62.39</td>
<td>10.23%</td>
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<td>305</td>
<td>1,030</td>
<td>725</td>
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<td>67.70</td>
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<td>1,800</td>
<td>1,495</td>
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<td>96.43</td>
<td>6.45%</td>
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<tr>
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<td>135.6</td>
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<tr>
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<td>41.55</td>
<td>8.39%</td>
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<tr>
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<td>610</td>
<td>46.9</td>
<td>7.68%</td>
<td>46.87</td>
<td>7.68%</td>
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<tr>
<td>420</td>
<td>1,800</td>
<td>1,380</td>
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<td>5.47%</td>
<td>75.60</td>
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<td>114.7</td>
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<td>4.36%</td>
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<td>1,800</td>
<td>1,190</td>
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<td>4.53%</td>
<td>53.91</td>
<td>4.53%</td>
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<td>3,050</td>
<td>2,440</td>
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<td>93.01</td>
<td>3.81%</td>
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<tr>
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<td>115</td>
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<td>4.64%</td>
<td>5.32</td>
<td>4.62%</td>
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<tr>
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<td>1,800</td>
<td>885</td>
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<td>3,050</td>
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<td>1,030</td>
<td>3,050</td>
<td>2,020</td>
<td>67.8</td>
<td>3.36%</td>
<td>67.83</td>
<td>3.36%</td>
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</table>
5. Case Study

### Performance of Reinsurance Layers \( \theta = 22.28 \)

<table>
<thead>
<tr>
<th>Retention (million)</th>
<th>Upper Limit (million)</th>
<th>Prob ( r &lt; 0 )</th>
<th>Prob ( r &lt; -15% )</th>
<th>Mean</th>
<th>Variance</th>
<th>Downside Variance</th>
<th>Risk-adjusted Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Reinsurance</td>
<td></td>
<td>18.41%</td>
<td>0.48%</td>
<td>3.916%</td>
<td>0.263%</td>
<td>0.070%</td>
<td>2.350%</td>
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<td>420</td>
<td>19.02%</td>
<td>0.42%</td>
<td>3.781%</td>
<td>0.253%</td>
<td>0.067%</td>
<td>2.291%</td>
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<td>610</td>
<td>19.17%</td>
<td>0.35%</td>
<td>3.771%</td>
<td>0.249%</td>
<td>0.064%</td>
<td>2.341%</td>
</tr>
<tr>
<td>610</td>
<td>915</td>
<td>19.31%</td>
<td>0.30%</td>
<td>3.779%</td>
<td>0.247%</td>
<td>0.061%</td>
<td>2.412%</td>
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<td>1,030</td>
<td>19.53%</td>
<td>0.27%</td>
<td>3.739%</td>
<td>0.243%</td>
<td>0.059%</td>
<td>2.428%</td>
</tr>
<tr>
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<td>1,800</td>
<td>19.95%</td>
<td>0.26%</td>
<td>3.676%</td>
<td>0.243%</td>
<td>0.057%</td>
<td>2.397%</td>
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<td>3,050</td>
<td>20.44%</td>
<td>0.41%</td>
<td>3.551%</td>
<td>0.247%</td>
<td>0.061%</td>
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<td>610</td>
<td>19.63%</td>
<td>0.33%</td>
<td>3.637%</td>
<td>0.241%</td>
<td>0.061%</td>
<td>2.268%</td>
</tr>
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<td>915</td>
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<td>0.25%</td>
<td>3.503%</td>
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<td>20.76%</td>
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<td>22.31%</td>
<td>0.13%</td>
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<td>0.210%</td>
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<td>2.231%</td>
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<td>0.04%</td>
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<td>0.042%</td>
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<td>915</td>
<td>19.85%</td>
<td>0.25%</td>
<td>3.634%</td>
<td>0.235%</td>
<td>0.057%</td>
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<td>0.232%</td>
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<td>21.79%</td>
<td>0.14%</td>
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<td>0.046%</td>
<td>2.330%</td>
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<td>24.25%</td>
<td>0.05%</td>
<td>2.995%</td>
<td>0.206%</td>
<td>0.043%</td>
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<td>0.16%</td>
<td>3.500%</td>
<td>0.226%</td>
<td>0.049%</td>
<td>2.402%</td>
</tr>
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<td>23.35%</td>
<td>0.11%</td>
<td>3.135%</td>
<td>0.215%</td>
<td>0.045%</td>
<td>2.124%</td>
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<td>915</td>
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<td>18.63%</td>
<td>0.40%</td>
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<td>0.067%</td>
<td>2.380%</td>
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<td>1,800</td>
<td>20.14%</td>
<td>0.21%</td>
<td>3.637%</td>
<td>0.239%</td>
<td>0.055%</td>
<td>2.407%</td>
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<td>915</td>
<td>3,050</td>
<td>22.44%</td>
<td>0.17%</td>
<td>3.272%</td>
<td>0.226%</td>
<td>0.050%</td>
<td>2.155%</td>
</tr>
<tr>
<td>1,030</td>
<td>3,050</td>
<td>22.15%</td>
<td>0.20%</td>
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<td>0.230%</td>
<td>0.052%</td>
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<td>680</td>
<td>1,390</td>
<td>20.00%</td>
<td>0.21%</td>
<td>3.667%</td>
<td>0.237%</td>
<td>0.055%</td>
<td>2.451%</td>
</tr>
</tbody>
</table>
5. Case Study

Efficient Frontier

Figure 3: Reinsurance Efficient Frontier
5. Case Study

Optimal Reinsurance Layers $\theta = 16.71, 22.28, 27.85$

<table>
<thead>
<tr>
<th>Theta</th>
<th>Retention (million)</th>
<th>Upper Limit (million)</th>
<th>Mean</th>
<th>Downside Variance</th>
<th>Risk-Adjusted Profit $\theta = 16.71$</th>
<th>Risk-Adjusted Profit $\theta = 22.28$</th>
<th>Risk-Adjusted Profit $\theta = 27.85$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16.71</td>
<td>795</td>
<td>1220</td>
<td>3.771%</td>
<td>0.060%</td>
<td>2.768%</td>
<td>2.434%</td>
<td>2.100%</td>
</tr>
<tr>
<td>22.28</td>
<td>680</td>
<td>1390</td>
<td>3.667%</td>
<td>0.055%</td>
<td>2.755%</td>
<td>2.451%</td>
<td>2.147%</td>
</tr>
<tr>
<td>27.85</td>
<td>615</td>
<td>1460</td>
<td>3.610%</td>
<td>0.052%</td>
<td>2.736%</td>
<td>2.445%</td>
<td>2.154%</td>
</tr>
</tbody>
</table>

If the overall profit rate increases 2% and $\theta$ remains at 22.28, the optimal layers becomes (740, 1420)
6. Conclusions

- The overall profitability (both cat and noncat losses) impacts optimal insurance decision
- Risk appetites are difficult to measure by a single parameter.
- DRAP capture risk appetites comprehensively though theta (risk aversion coefficient), T (downside benchmark), and moment k (increasingly perception toward large loss)
- DRAP provides an alternative approach to calculate optimal layers.

Q & A